Parametrix of Static Hedge (of a Timing Risk)

Jirô Akahori

Ritsumeikan University

Dec. 1, 2014

joint work with Yuri Imamura[1] and Flavia Barsotti [2]

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[1] Ritsumeikan University

[2] Risk Methodologies, Group Financial Risks, UniCredit SpA, Milan.

The views presented in this paper are solely those of the author and do not necessarily represent those of UniCredit Spa.

Overview, Parametrix?

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- In the latter, it is known that a perfect static replication of a barrier option is possible. (P. Carr,...)
- I will introduce a **parametrix** of barrier type options in a general diffusion environment by the static hedge in BS.

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- Barrier options, defaultable bonds, or American options have a timing risk.
- European claims <u>do not have</u> any timing risk since the payment occurs only at the prescribed time.

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- We do not want to take a position with a timing risk. We rather want to exchange the position to the one without a timing risk.
- "Without a timing risk" means a portfolio composed only of underlyings and European type options. We call such an exchange technique **semi-static hedge**.

Static Hedge of a knock out option [under BS]

It has been widely known since the paper by P. Carr and J. Bowie (1994) that under Black-Scholes assumptions, simple Barrier options can be hedged by a static position of two options.

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- Start with long position of a call option and a short position of a put.
- When the boundary is hit, the value of the two option coincides (by the BS assumption) and so we can liquidate the position with no extra cost.
- If the boundary is never hit until the maturity, the call option at hand hedges the barrier option.

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When the option to be hedged is a call option knocked in when the underlying hits a boundary, it can be hedged by

- Start with long position of a put.
- When the boundary is hit, the value of put option coincides (by the BS assumption) with that of call option and so we can exchange them with no extra cost.

Static Hedge of a knock IN option under BS

When the option to be hedged is a call option knocked in when the underlying hits a boundary, it can be hedged by

- Start with long position of a put.
- When the boundary is hit, the value of put option coincides (by the BS assumption) with that of call option and so we can exchange them with no extra cost.
- If the boundary is never hit until the maturity, both the knock-in option and the put option at hand are worth zero.

Static Hedge of the simplest timing risk

P. Carr and J. Picron (1999), under a Black-Scholes environment, tried to apply the semi-static hedging formula of barrier options to hedge a constant payment at a stopping time (which actually is a hitting time).

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- P. Carr and J. Picron found that an integration of the semi-static hedging formula for barrier options provides a semi-static hedge of the timing risk of constant payment.
- Under Black-Scholes economy, the integral of the semi-static hedging formula of barrier options of Bowie and Carr type provides a **perfect** static hedge of the timing risk.
- The integral (with respect to maturities) implies that the static hedging portfolio consists of (infinitesimal amount of) options with different (continuum of) maturities, which should be discretized in practice.

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- This procedure gives a (Taylor-like) series expansion of semi-static hedge of a timing risk.
- This is our **Parametrix**.

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We will work on a filtered probabilty space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$, and ...etc.

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Definition

Let $T > 0$ be fixed. Let Θ be an $\mathcal{F}_T$-measurable random variable. Another $\mathcal{F}_T$-measurable random variable Θ' is called in (abstract) Put-Call Symmetry of Θ with respect to (X, D) if

$$1_{\{\tau<T\}}E[\Theta 1_{\{X\in D\}}|\mathcal{F}_\tau] = 1_{\{\tau<T\}}E[\Theta' 1_{\{X\notin D\}}|\mathcal{F}_\tau]. \qquad (1)$$

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If it is the case, the knock option whose pay-off at maturity T is $\Theta 1_{\{X\in D\}}1_{\{\tau>T\}}$, is hedged by long of $\Theta 1_{\{X\in D\}}$ and short of $\Theta' 1_{\{X\notin D\}}$ of non-knock out options since...

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- A put option $(K - X_T)_+$ is in put-call symmetry of call option $(X_T - K)_+$ with respect to the region $x > K$ when $r = 0$.
- More generally, $(\frac{X_T}{K})^{1-\frac{2r}{\sigma^2}} f(\frac{K^2}{X_T})$ is in put-call symmetry of $f(X_T)$.

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We have that

$$\begin{aligned}
\mathbf{E}[e^{-r\tau}] &= \int_0^\infty e^{-rt}\mathbf{P}(\tau \in dt) \\
&= [e^{-rt}\mathbf{P}(\tau < t)]_0^\infty + r\int_0^\infty e^{-rt}\mathbf{P}(\tau < t)dt \\
&= r\int_0^\infty e^{-rt}\mathbf{P}(\tau < t)dt \\
&= r\int_0^\infty e^{-rt}(1 - \mathbf{E}[I_{\{\tau > t\}}])dt \\
&= r\int_0^\infty e^{-rt}\mathbf{E}[(1 + (\frac{X_t}{K})^{1-\frac{2r}{\sigma^2}})I_{\{X_t \le K\}}|\mathcal{F}_\tau]dt.
\end{aligned}$$

Carr and Picron's static hedge

In short,

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Note that

- The timing risk is decomposed into the integral of the digital knock-in options with exercise price K and maturity $t \in (0, \infty)$ with the volume rdt.

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- The timing risk is decomposed into the integral of the digital knock-in options with exercise price K and maturity $t\in(0,\infty)$ with the volume rdt.
- With a put-call symmetry, the knock-in option is rewritten as options without timing risk.

Imperfect Semi-Static Hedge

Let us consider the static hedge a knock-out option with pay-off Θ **without put-call symmetry**.

The hedge is:

- buys a European option with pay-off $\Theta 1_{\{X_T \in D\}}$,
- sells a European option with pay-off $\Theta' 1_{\{X_T \notin D\}}$.

The value of the portfolio at time t is :

$$e^{-r(T-t)}\mathbf{E}[\Theta 1_{\{X_T \in D\}} - \Theta' 1_{\{X_T \notin D\}} | \mathcal{F}_t].$$

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Hedging Error

$$\begin{aligned} Err_t &:= -e^{-r(T-t)}\mathbf{E}[\Theta 1_{\{\tau > T\}} | \mathcal{F}_t] \\ &\qquad + e^{-r(T-t)}\mathbf{E}[\Theta 1_{\{X_T \in D\}} - \Theta' 1_{\{X_T \notin D\}} | \mathcal{F}_t] \\ &= e^{-r(T-t)}\mathbf{E}[1_{\{\tau \leq T\}} \left(\Theta 1_{\{X_T \in D\}} - \Theta' 1_{\{X_T \notin D\}} \right) | \mathcal{F}_t]. \end{aligned}$$

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Put-Call Symmetry $\Rightarrow Err_t = 0$

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From now on I will explain why and how the error is represented as an integral of knock-in options, just as Carr-Picron's, when $\Theta = f(X_T)$, and $\Theta' = (\pi f)(X'_T)$ is in put-call symmetry of $f(X'_T)$ with respect to another diffusion process X'.

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As announced, the Paramatrix is a key ingredient.

Parametrix

Let p_t and q_t be transition densities (assuming they exist) of process Y and Y', respectively. We consider a "parametrix" of q_t by p_t using the following relation:

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Lemma

It holds that

$$q_t(x,y) - p_t(x,y) = \int_0^t ds \int_{\mathbb{R}^d} dz\ q_s(x,z)(L_z - L'_z)p_{t-s}(z,y)$$

where $L = L_x$ and $L' = L'_x$ denote the infinitesimal generator of Y and Y', respectively, acting on the variable x.

Parametrix, proof of the lemma

(Proof) We have that

$$\begin{aligned}\partial_s\{q_s(x,z)p_{t-s}(z,y)\} &= \partial_s q_s(x,z)p_{t-s}(z,y) - q_s(x,z)\partial_s q_{t-s}(x,z)\\ &= (L_z^* q_s)(x,z)p_{t-s}(z,y) - q_s(x,z)(L_z' p_{t-s})(z,y),\end{aligned}$$

where L^* denotes the adjoint operator of L.

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where L^* denotes the adjoint operator of L.
By integrating the equation, we obtain the RHS of the lemma;

$$\begin{aligned}&\lim_{\epsilon\downarrow 0}\int_\epsilon^{t-\epsilon}\partial_s\{q_s(x,z)p_{t-s}(z,y)\}ds\\ &=\lim_{\epsilon\downarrow 0}\int_\epsilon^{t-\epsilon} ds\int_{\mathbb{R}^d} dz\{(L_z^* q_s)(x,z)p_{t-s}(z,y) - q_s(x,z)(L_z^y p_{t-s})(z,y)\}\\ &=\int_0^t ds\int dz\ q_s(x,z)(L_z - L_z^y)p_{t-s}(z,y).\end{aligned}$$

Parametrix, proof of the lemma

On the other hand, since

$$\lim_{s \downarrow 0} \int q_s(x,z) p_{t-s}(z,y)\, dz = p_t(x,y)$$

and

$$\lim_{s \uparrow t} \int q_s(x,z) p_{t-s}(z,y)\, dz = q_t(x,y),$$

we obtain the LHS;

$$\lim_{\epsilon \downarrow 0} \int_{\epsilon}^{t-\epsilon} \partial_s \{ q_s(x,z) p_{t-s}(z,y) \} ds = q_t(x,y) - p_t(x,y).$$

q.e.d.

Expression of the Error

Now we are back in the specific situation where $Y = X$ and $Y' = X'$, etc. We put, for $y \in D$,

$$h_0(t, z, y) := (L_z - L'_z)p_t(z, y),$$

and

$$h(t, z; y) := h_0(t, z, y) - \pi^*_y h_0(t, z, y),$$

where π^*_y is the adjoint of π acting on y.

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where π^*_y is the adjoint of π acting on y. Define a one-parameter family of integral operators $\{S(t)\}$ by

$$S_t f(x) = \int_D h(t, x, y) f(y)\, dy, \quad t \geq 0.$$

Expression of the Error

Then, we have the following

Theorem

The error is equal to the value of the integral of knock-in options with pay-off $S_{T-s}f(X_s)ds$;

$$\mathrm{Err}_t = e^{-r(T-t)}E[\int_t^T 1_{\{\tau \leq s\}} S_{T-s} f(X_s)\, ds | \mathcal{F}_t].$$

Expression of the Error, proof of the theorem

(Proof) Observe that

$$\mathrm{Err}_t = e^{-r(T-t)}E[1_{\{\tau \le T\}}\{f(X_T)1_{\{X_T\in D\}} - \pi f(X_T)1_{\{X_T\notin D\}}\}|\mathcal{F}_t]$$
$$= e^{-r(T-t)}E[1_{\{\tau \le T\}}E[\{f(X_T)1_{\{X_T\in D\}} - \pi f(X_T)1_{\{X_T\notin D\}}\}|\mathcal{F}_\tau]|\mathcal{F}_t].$$

Thanks to the optional sampling theorem, the expectation conditioned by $\mathcal{F}_\tau$ turns into

$$\int_D f(y)q_{T-\tau}(X_\tau, y)\,dy - \int_{D^c} \pi f(y)q_{T-\tau}(X_\tau, y)\,dy$$

on the set $\{\tau \le T\}$.

Expression of the Error, proof of the theorem

By applying the lemma (of parametrix), we have

$$\int_D f(y) q_{T-\tau}(X_\tau, y)\, dy = \int_D f(y) p_{T-\tau}(X_\tau, y)\, dy$$
$$+ \int_\tau^T ds \int_{\mathbb{R}^d} dz\ q_{s-\tau}(X_\tau, z) \int_D h_0(T-s, z, y) f(y)\, dy,$$

Expression of the Error, proof of the theorem

which is also valid for πf;

$$\int_{D^c} \pi f(y) q_{T-\tau}(X_\tau, y)\, dy = \int_{D^c} \pi f(y) p_{T-\tau}(X_\tau, y)\, dy$$
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Since πf is in pcs of f w.r.t. p_t, we know that

$$\int_{D^c} \pi f(y) p_{T-\tau}(X_\tau, y)\, dy = \int_D f(y) p_{T-\tau}(X_\tau, y)\, dy.$$

Expression of the Error, proof of the theorem

Now we see that the expectation conditioned by $\mathcal{F}_\tau$

$$\int_D f(y) q_{T-\tau}(X_\tau, y)\, dy - \int_{D^c} \pi f(y) q_{T-\tau}(X_\tau, y)\, dy$$

is equal to

$$\begin{aligned}
&\int_\tau^T ds \int_{\mathbb{R}^d} dz\ q_{s-\tau}(X_\tau, z) \\
&\cdot \left\{ \int_D h_0(T-s, z, y) f(y)\, dy - \int_{D^c} h_0(T-s, z, y) \pi f(y)\, dy \right\} \\
&= \int_\tau^T ds \int_{\mathbb{R}^d} dz\ q_{s-\tau}(X_\tau, z) \\
&\cdot \int_D \left\{ h_0(T-s, z, y) - \pi_y^* h_0(T-s, z, y) \right\} f(y)\, dy \\
&= \int_\tau^T ds \int_{\mathbb{R}^d} dz\ q_{s-\tau}(X_\tau, z) S_{T-s} f(z).
\end{aligned}$$

Expression of the Error, proof of the theorem

Noting that

$$\int_{\mathbb{R}^d} dz\ q_{s-\tau}(X_\tau, z) S_{T-s} f(z) = E[1_{\{\tau \le s\}} S_{T-s} f(X_s) | \mathcal{F}_\tau],$$

we have

$$\begin{aligned} \mathrm{Err}_t &= e^{-r(T-t)} E[1_{\{\tau \le T\}} \int_\tau^T E[1_{\{\tau \le s\}} S_{T-s} f(X_s) | \mathcal{F}_\tau] \, ds | \mathcal{F}_t] \\ &= e^{-r(T-t)} \int_t^T E[1_{\{\tau \le s\}} S_{T-s} f(X_s) | \mathcal{F}_t] \, ds. \end{aligned}$$

□

Second Order Static Hedge

The formula of the previous theorem

$$\mathrm{Err}_t = e^{-r(T-t)} \int_t^T E[S_{T-s} f(X_s) - 1_{\{\tau \geq s\}} S_{T-s} f(X_s) | \mathcal{F}_t] \, ds,$$

claims that the Err_t can be understood as a timing risk. We set

$$\eta g(x) := g(x) 1_{\{x \in D\}} - \pi g(x) 1_{\{x \notin D\}}.$$

Then Err_t is hedged by a portfolio composed of options with pay-off

$$(1-\eta) S_{T-s} f(X_s) = \{S_{T-s} f(X_s) + \pi S_{T-s} f(X_s)\} 1_{\{X_s \notin D\}},$$

and the volume "$e^{-r(T-s)} ds$".

Note that the value at time t of the portfolio is given by

$$e^{-r(T-t)} E[\int_t^T (1-\pi) S_{T-s} f(X_s) ds | \mathcal{F}_t] ds.$$

Second Order Static Hedge

The hedge error coincides with the one in the corresponding knock-out case. So the error of the static hedge for the option maturing s is given by

$$\mathrm{Err}^s_{2,t} ds := e^{-r(T-t)} E[1_{\{\tau \le s\}} \pi S_{T-s} f(X_s) | \mathcal{F}_t] ds.$$

Now we can apply Theorem to obtain that

$$\mathrm{Err}^s_{2,t} ds = e^{-r(T-t)} \int_t^s E[1_{\{\tau \le u\}} S_{s-u} S_{T-s} f(X_u) | \mathcal{F}_t] \, duds,$$

We know that $\mathrm{Err}^s_{2,t}$ is integrable in s on $[t, T]$ almost surely. Thus the totality of the error is obtained as

$$\int_t^T \mathrm{Err}^s_{2,t} ds = e^{-r(T-t)} \int_t^T ds \int_t^s du E[1_{\{\tau \le u\}} S_{s-u} S_{T-s} f(X_u) | \mathcal{F}_t]$$
$$=: \mathrm{Err}_{2,t}.$$

n-th Order Static Hedge

By repeating this procedure, we obtain the n-th order static hedges and the n-th error for any n:

Theorem

We have that

$$e^{-r(T-t)}\Big\{ - E[f(X_T)1_{\{\tau>T\}}|\mathcal{F}_t] + E[\pi f(X_T)|\mathcal{F}_t] - \sum_{k=1}^{n}\int_t^T ds\ E[(1-\eta)S^{*k}_{T-s}(X_s)|\mathcal{F}_t]\Big\} = e^{-r(T-t)}\int_t^T du\ E[1_{\{\tau\le u\}}S^{*n}_{T-u}f(X_u)|\mathcal{F}_t] =: \mathrm{Err}_{n,t},$$

where

$$S^{*1}_t = S_t, \quad S^{*k}_t = \int_0^t S_s S^{*(k-1)}_{t-s} ds, k = 2, 3, \cdots .$$

Perfect Static Hedge

The previous formula reads: the semi-static hedge of the knock-out option with pay-off f, by the option with pay-off $\eta f(X_T)$ and the options with pay-off $\sum_{k=1}^{n}(1-\eta)S^{*k}_{T-u}f(X_u)du$ for $u < T$, has the error

$$\int_t^T du\ E[S^{*n}_{T-u}f(X_u)|\mathcal{F}_\tau] \tag{2}$$

at the knock-out time τ.

Perfect Static Hedge

Under suitable conditions, it converges to zero as n goes infinity so that we have the following

Theorem

The series $\sum_{k=1}^{\infty}(1-\eta)S_{T-u}^{*k}f(X_u)$ *is absolutely convergent almost surely, and the option with pay-off* $\eta f(X_T)$ *and the options with pay-off* $\sum_{k=1}^{\infty}(1-\eta)S_{T-u}^{*k}f(X_u)e^{-r(T-u)}du$ *for each* $u < T$ *gives a perfect semi-static hedge (the error is zero almost surely).*

Hint

Hint: the lemma can be rephrased as

$$q_t(x,y) = p_t(x,y) + \int_0^t ds \int dz\, q_s(x,z) h_0(z,y)$$

by which q can be understood as a solution to a Volterra type equation.

Thank you for your kind attentions.