Dimensionality as a measure of financial market and portfolio health

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Standard Life Investments

Diversification disaster

Hot topic with no definition

- No formula for diversification – unlike volatility
- Possibilities
  - Number of shares in your portfolio
  - Balance of contributions to portfolio risk
  - Resilience to stress testing
- Lack of definitive measure limits ones ability to monitor and optimise

Beyond portfolios

- Traditional to think about portfolio diversification
  - Expected returns
  - Expected covariances
  - Efficient frontiers
- Diversification potential within an investment universe
  - How many investment opportunities are available?
  - How many named factors can I get exposure to?
  - How many statistical factors explain how much risk?

Thinking about market health

- Diversification potential has a natural link to market health
- Healthy markets allow us to build differentiated portfolios
- Diverse opportunities link to active two-way trading and liquidity
- Concept is well established in Competition and Antitrust law

Diversification potential and monopolies

- Herfindahl-Hirschman Index (HHI) for concentration within industries
- Given market shares P_i s. t. $\sum P_i = 1$, HHI = $\sum {P_i}^2$

P = {1,0,…,0} HHI = 1 => most concentrated

$P = \{\frac{1}{10}, \frac{1}{10}, \ldots, \frac{1}{10}\}$ HHI = $\frac{1}{10}$ => most unconcentrated

"The agencies generally consider markets in which the HHI is between 0.15 and 0.25 to be moderately concentrated, and consider markets in which the HHI is in excess of 0.25 to be highly concentrated." *http://www.justice.gov/atr/public/guidelines/hhi.html*

Practical with limitations

- M&A activity in moderate and highly concentrated markets under scrutiny
- Guideline *large* change for HHI is between 0.01 and 0.02
- > 0.02 presumed to be likely to enhance market power
- Useful but perhaps not so intuitive

Hill's proposals

- A more intuitive measure with a sensible range $1/HHI$
- But does this make sense? What is the motivation?
- Hill (1973) introduced effective numbers based on Renyi entropies

$$D^{\alpha} = \left(\sum_{i}^{n} {P_i}^{\alpha}\right)^{\frac{1}{1-\alpha}} = \exp(H_{\alpha}(P))$$

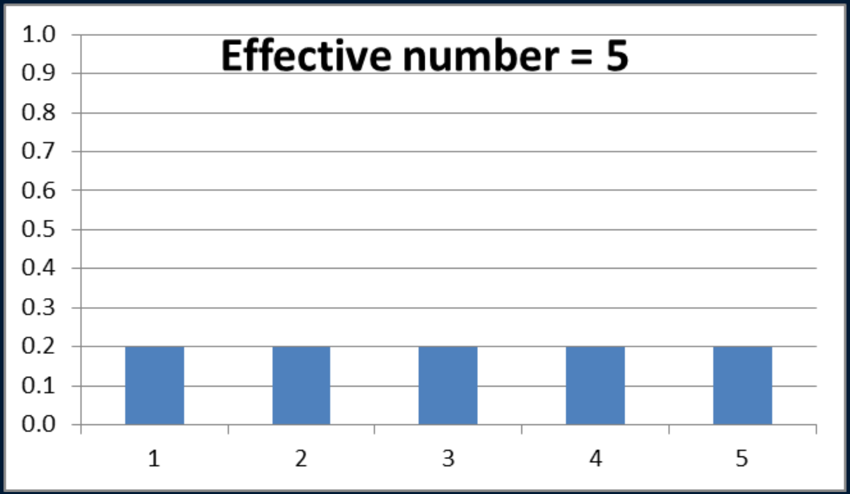
- where $H_{\alpha}(P)$ is the Renyi entropy of order $\alpha \geq 0$: $H_{\alpha}(P) = \frac{1}{1-\alpha}\ln(\sum {P_i}^{\alpha})$

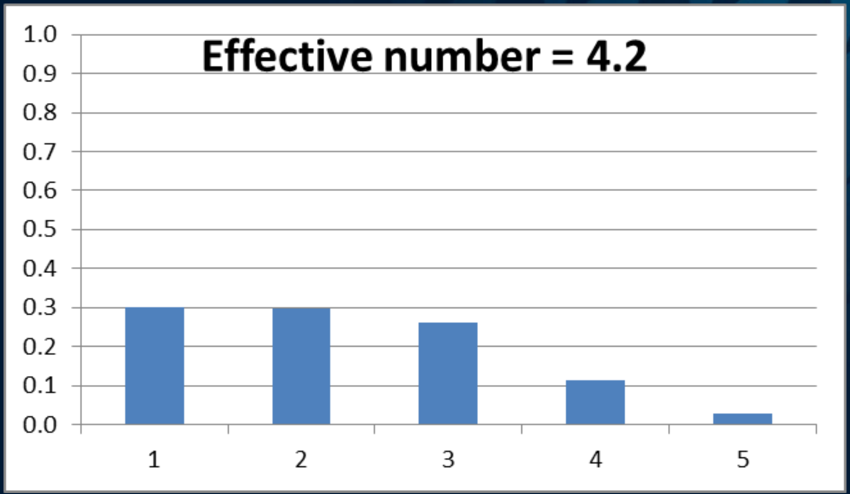
Spectrum of α

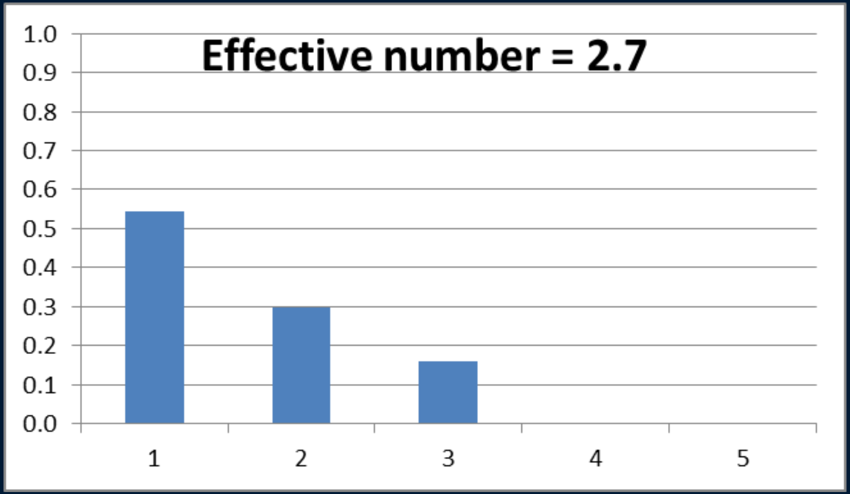
$\boldsymbol{\alpha}$	D^{α}	H_{α}
0	$D^0 = n$	Max or Hartley
1	$D^1 = \exp(H(P))$	Shannon
2	$D^2 = \left(\sum {P_i}^2\right)^{-1}$ = 1/ HHI	Collision
3	$D^3 = \left(\sum {P_i}^3\right)^{-1/2}$ =	…
…	…	…
∞	$D^{\infty} = {}^{1}/_{\sup\{P_i\}}$	Min

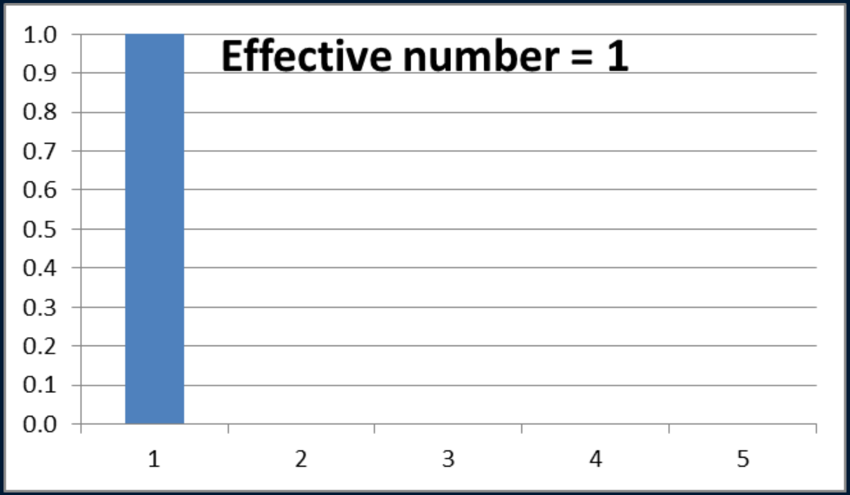
- Where the Shannon entropy $H(P) = H_1(P) = -\sum P_i \ln P_i$
- For each $\alpha > 0$, D^{α} ranges between *1* and *n* and is a measure of *flatness*

Ubiquity of Shannon - D^1









Related concepts and applications

- $\exp(H(P))$ as effective support size (ESS) of a distribution Grendar (2006)
  - Support up to 5 above, but effective support smaller
  - Generalisations based on Renyi possible, but not preferred
  - Relevant for continuous distributions and differential entropy

- Effective alphabet size Cover (2006)
  - Compressing data with character frequencies P_i

- Effective numbers of species as diversity in an ecosystem Jost (2006)
  - Frequency of species P_i: HHI $\equiv$ Simpson Index
  - Entropy is an index of diversity but not *the* measure
  - Hill numbers transform entropy into units of species

Dimensionality as industry health

- Effective (Hill) numbers provide a measure of effective dimensionality
  - Higher dimensionality => more companies => better competition
  - More intuitive than HHI

- Fails to account for degree of similarity and difference
  - HHI: Are products pure substitutes? How to measure market share? Share of industry sales? Domestic only? Total capacity? Collusion?
  - Simpson: genetic similarity of species in ecosystems

- Company concentration interesting measure of health *but not in crises*

Price volatility and correlation disaster

Dimensionality as financial market health

- Similarity in price movements may be a good measure – how?
- Average pair-wise correlation of stocks/industries not truly multi-dimensional
- Obvious link to index volatility and MPT through the covariance matrix Σ
- If healthy markets are higher dimensional what about rank(Σ)?

Matrix rank

- Useful concept for integer matrices but rarely for real data

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0.99 & 0.98 \\ 0.99 & 1 & 0.99 \\ 0.98 & 0.99 & 1 \end{pmatrix}$$

- What are the ranks? What do you think they should be?

- Effective rank of Roy and Vetterli (2007)

  - PCA for normalised eigenvalue spectrum $P_i = {}^{\lambda_i}/_{\sum \lambda_i}$

  - erank(A) = $\mathrm{ESS}(\mathrm{P}) = \exp(H(P))$

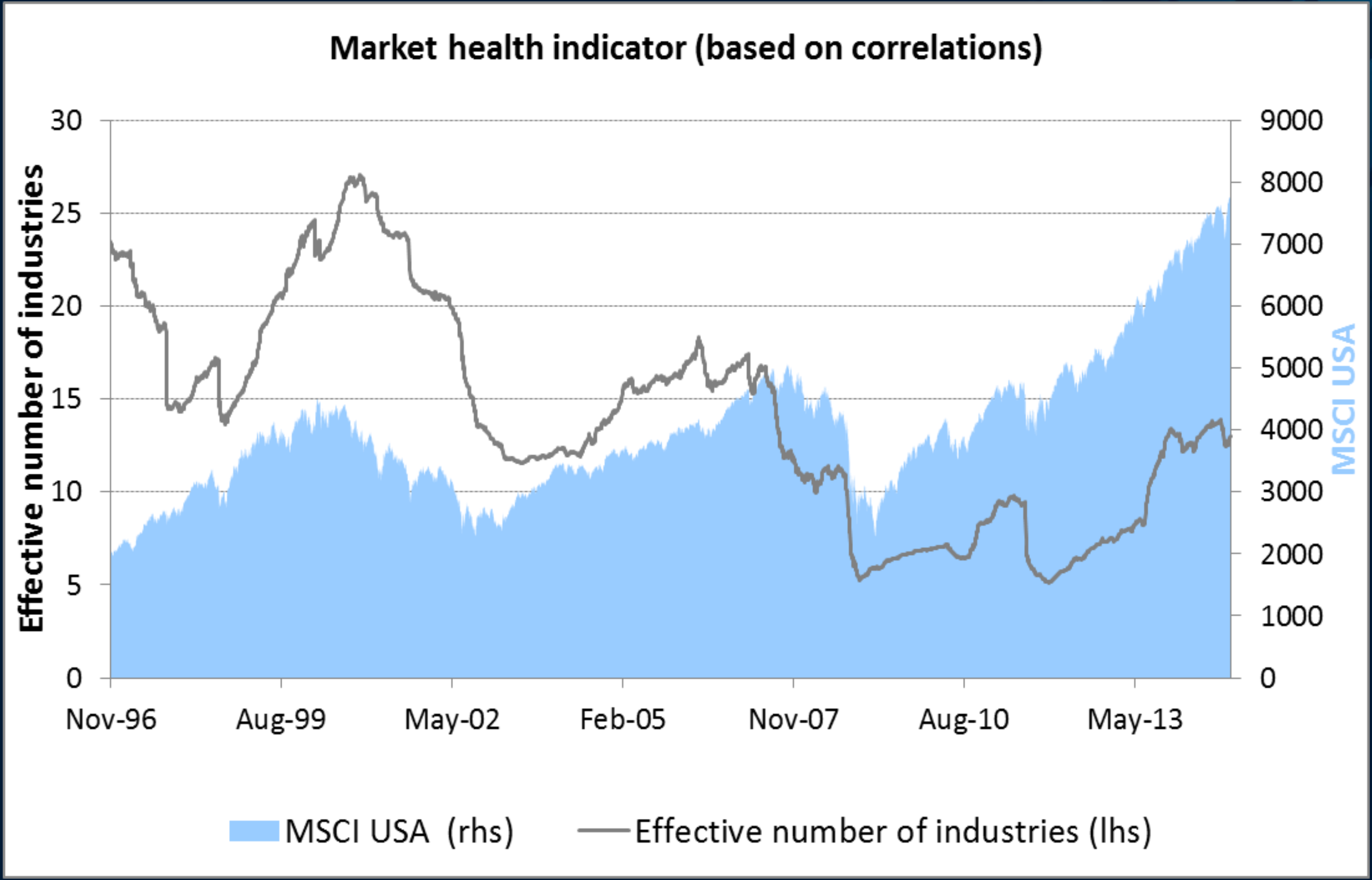
A measure of effective dimensionality

- A continuous version of matrix rank
  - Property 1: $1 \leq \mathrm{erank}(A) \leq \mathrm{rank}(A) \leq n$
  - Property 2: $\mathrm{erank}(A) = \mathrm{erank}(A^*) = \mathrm{erank}(A^T)$
  - Property 3: a unitary transform leaves erank unchanged
  - Property 4: $\mathrm{erank}(A + B) \leq \mathrm{erank}(A) + \mathrm{erank}(B)$

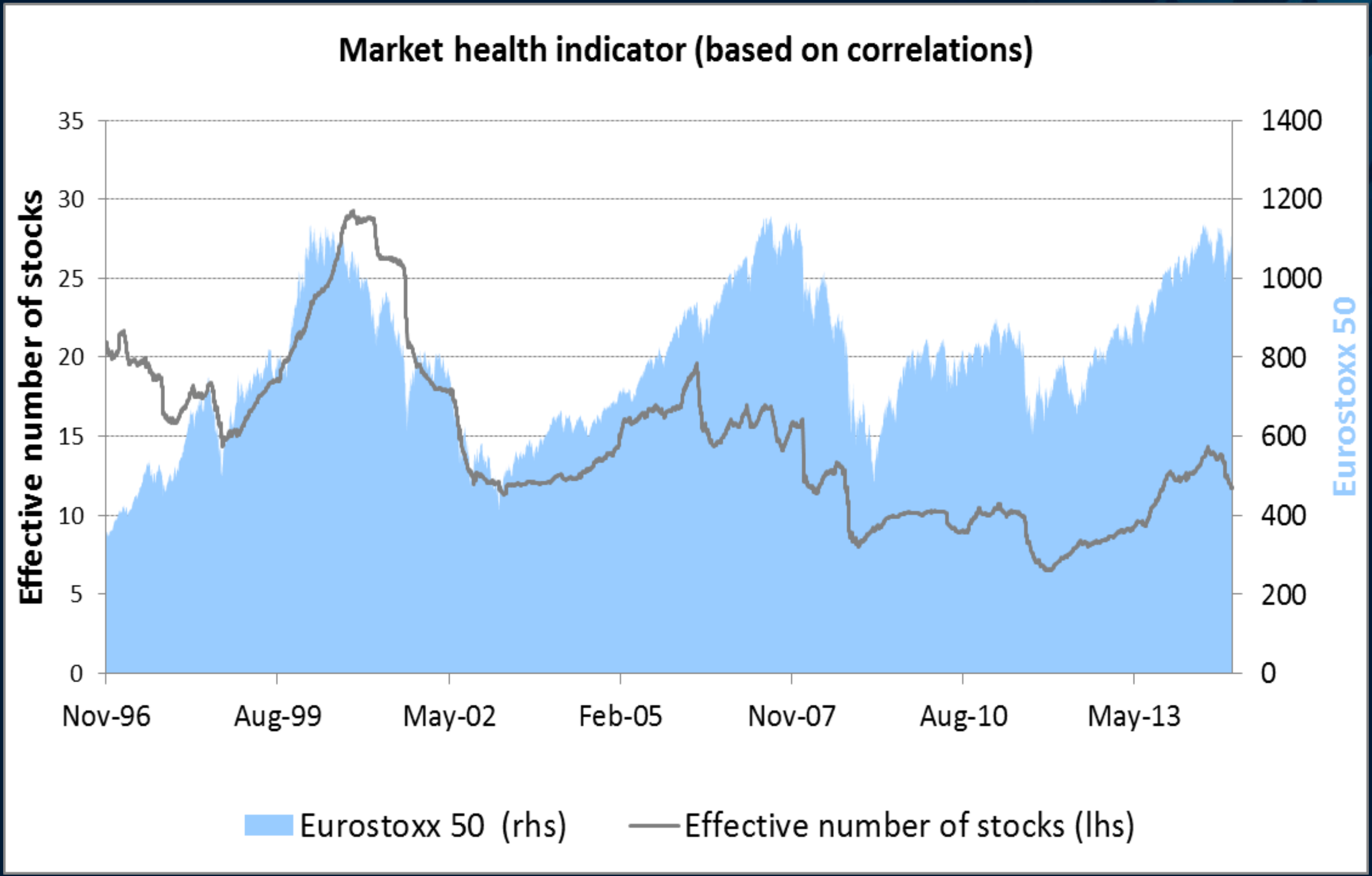
- A continuous solution to low-rank matrix approximation

- Matrix C $P = \{99.11\%, 0.66\%, 0.22\%\}$ erank = 1.06

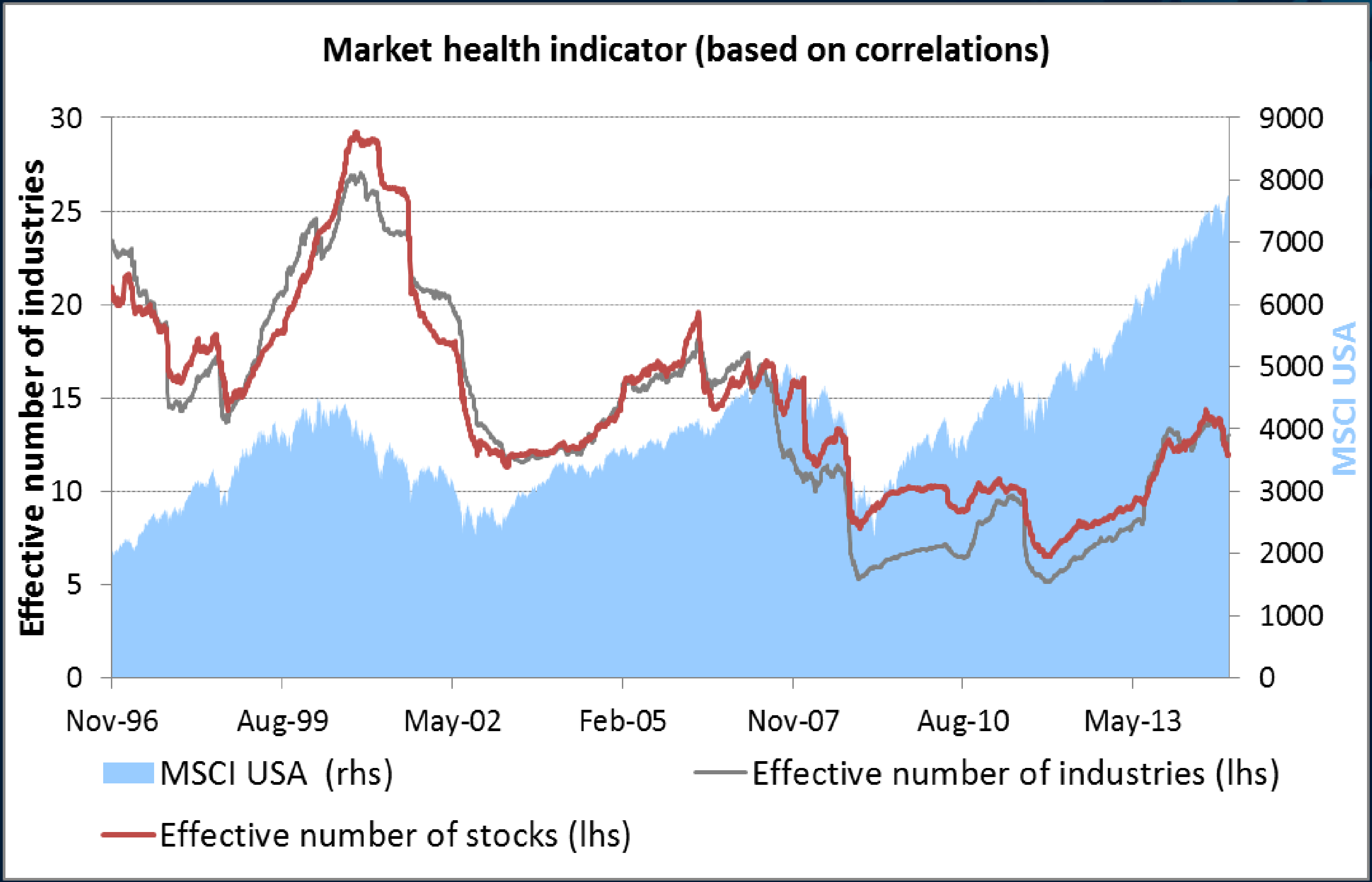
Dimensionality of the MSCI US index



Dimensionality of the Euro Stoxx 50 index



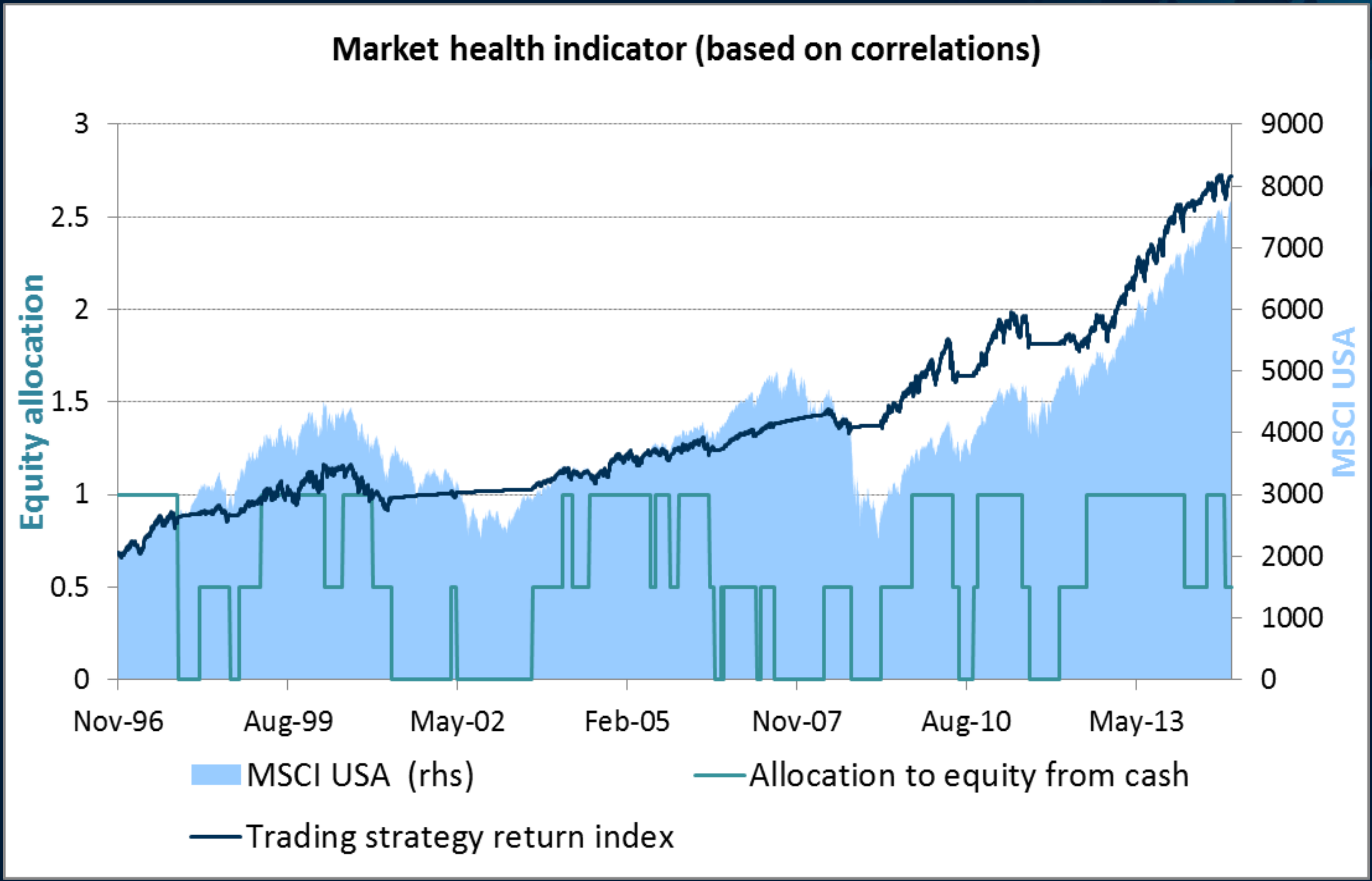
Commonality of dimensionality



How might this be used?

- Absorption ratio (AR) of Kritzman (2011) produces very similar indicator
- AR $\equiv$ % of variance explained by top 20% of eigenvalues of Σ
- Sharp increase in AR followed by persistence underperformance
- Simple trading rule can add value in asset allocation

Trading the dimensionality

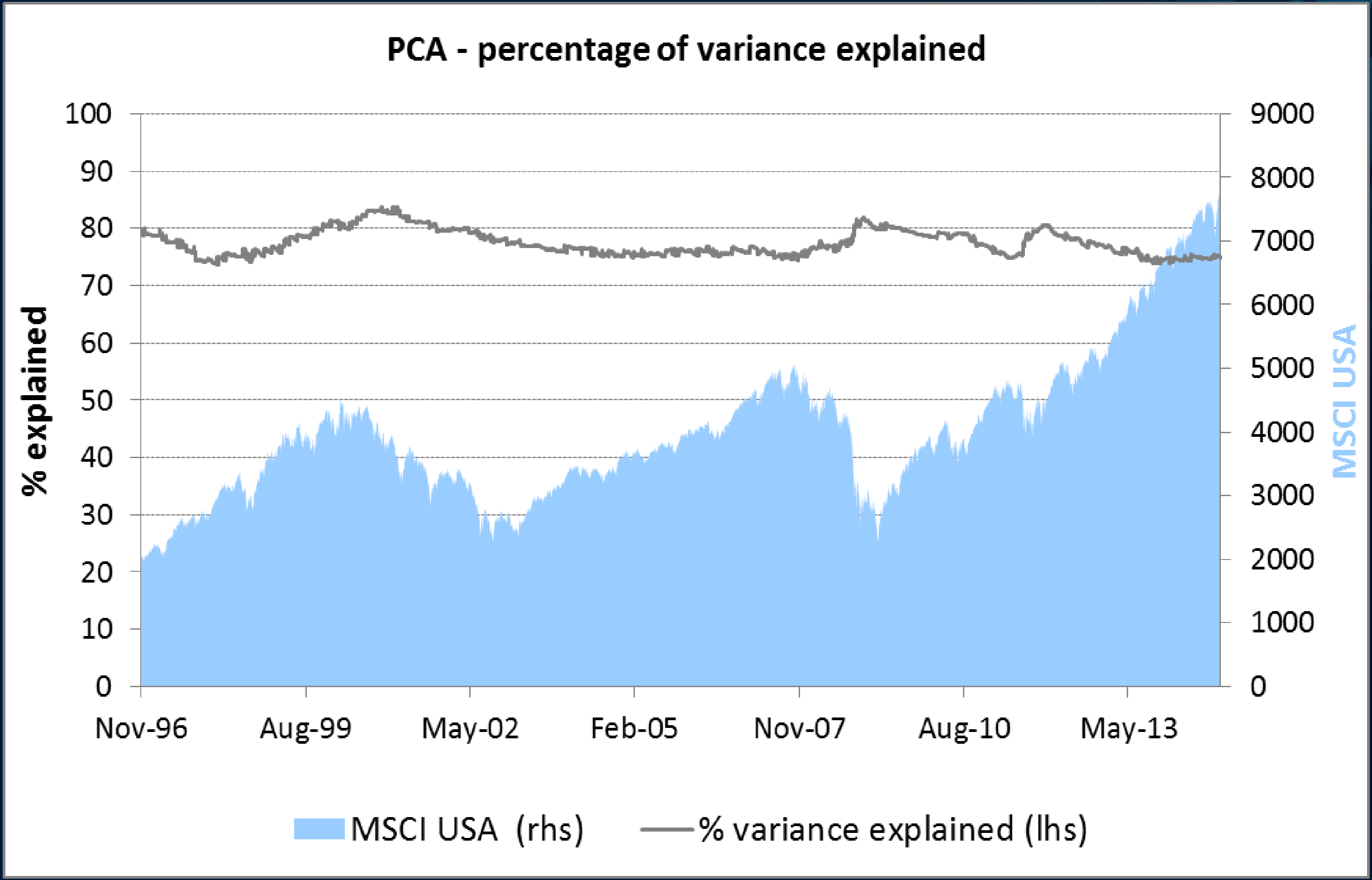


Comparison of effective rank over AR

- AR focused on top 20% of eigenvalues
  - Good historically for trading signal
  - 20% cut-off works for larger *n*
  - Truncates the eigenvalue spectrum and avoids noise

- Effective rank
  - Intuitive number that works smoothly for any number of time series
  - Includes the full eigenvalue spectrum - so no cutoff
  - Explains a stable proportion of total variance – Fleming/Kroeske (2013)

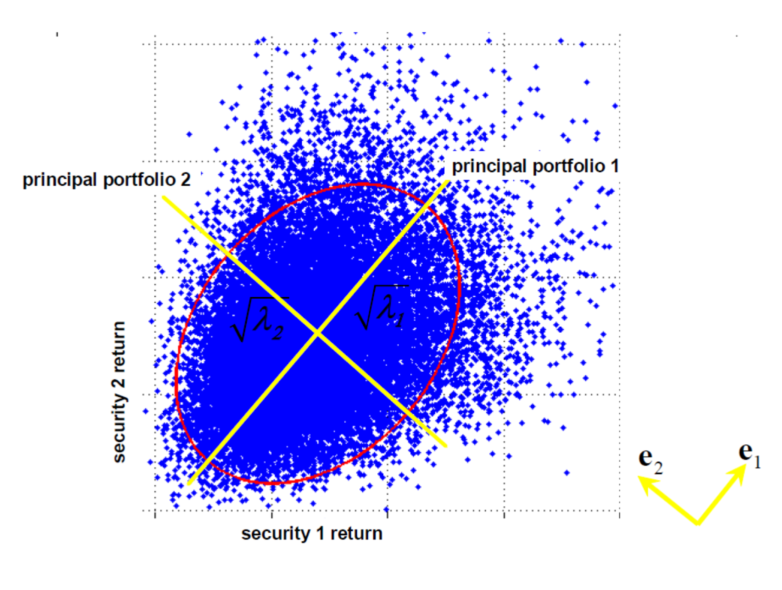
Stable variance explained



Dimensionality as portfolio health

- Remember that Σ is not the market portfolio
  - PCA concerned with $\mathrm{Tr}(\Sigma) = \sum \sigma_{ii} = \sum \lambda_i$
  - Portfolio risk: ${\sigma_p}^2 = w'\Sigma w = \sum_{i,j} w_i w_j \sigma_{ij}$

Meucci (2009)



Effective number of bets

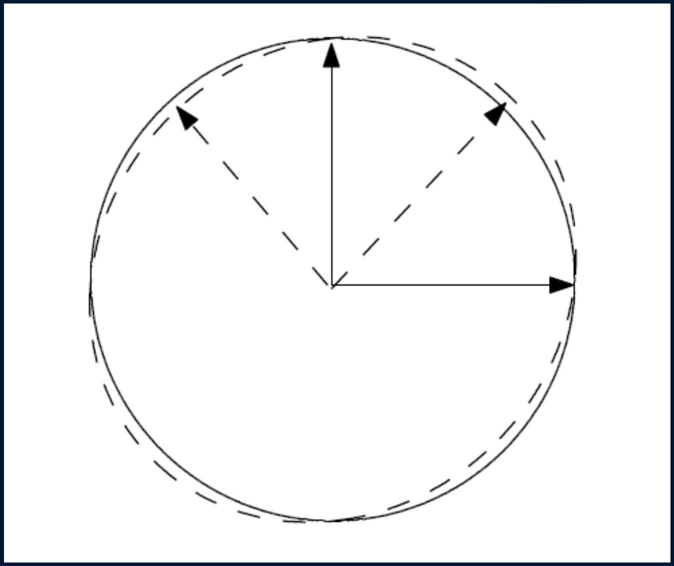
- Meucci (2009) introduces effective number of bets (or assets)
- Use PCs to decompose portfolio risk into uncorrelated proportions

$$\sigma_p{}^2 = \sum_1^n \varpi_i{}^2 \, \lambda_i \quad \Rightarrow \quad p_i = {}^{\varpi_i{}^2 \, \lambda_i}\!\big/_{\sigma_p{}^2}$$

- p_i is the proportion of portfolio variance explained by principal portfolio *i*
- $\exp(H(P))$ defines the effective number of bets between *1* and *n*

Eigenvalues versus eigenvectors

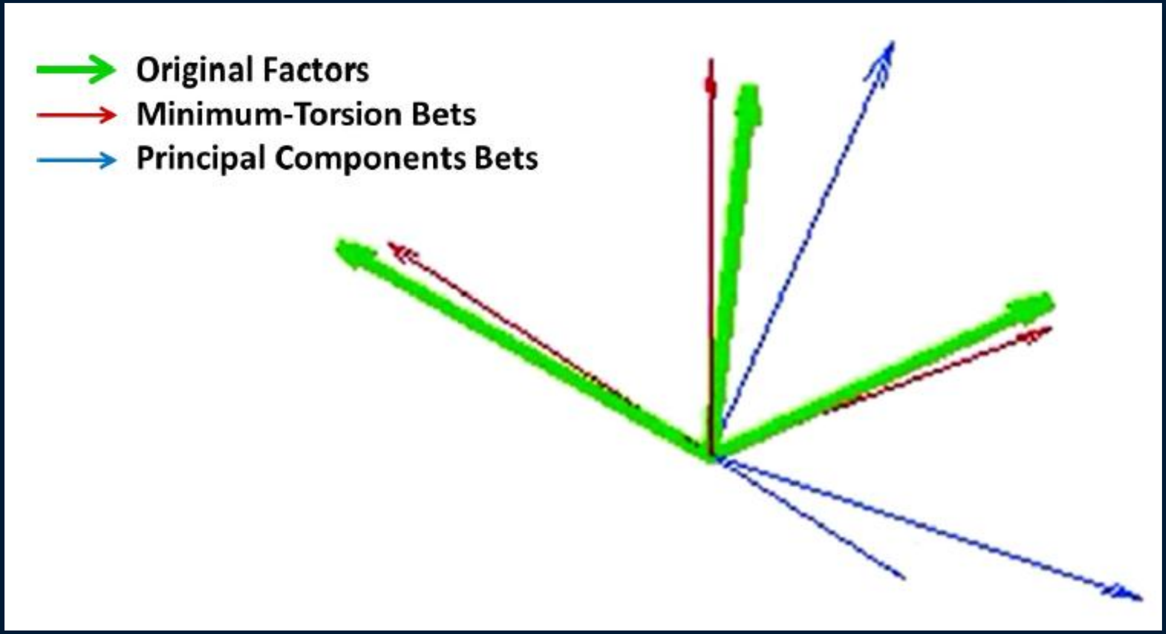
- Market health indicator a function of eigenvalues only
- Eigenvectors (and principal portfolios) inherently less stable
  - Undefined in perfect state with counterintuitive results
  - Undefined with eigenvalue multiplicity
  - Arbitrary sign convention - 2^n solutions
- Updated by Meucci (2013)



Ng et al. (2002)

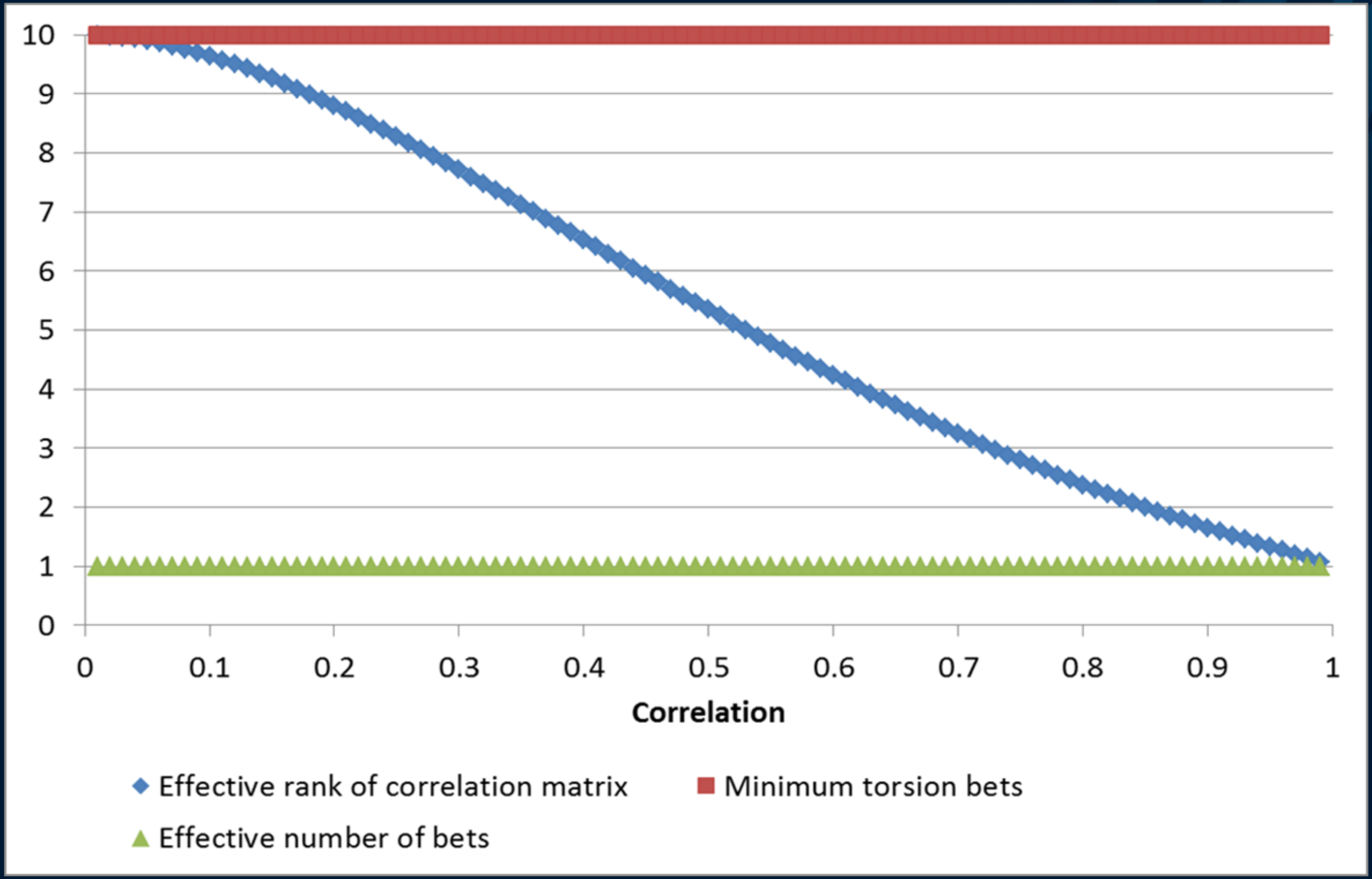
Minimum torsion bets

- Choose a rotation that is closest to the original axes
- But does this make sense?



Meucci et al. (2013)

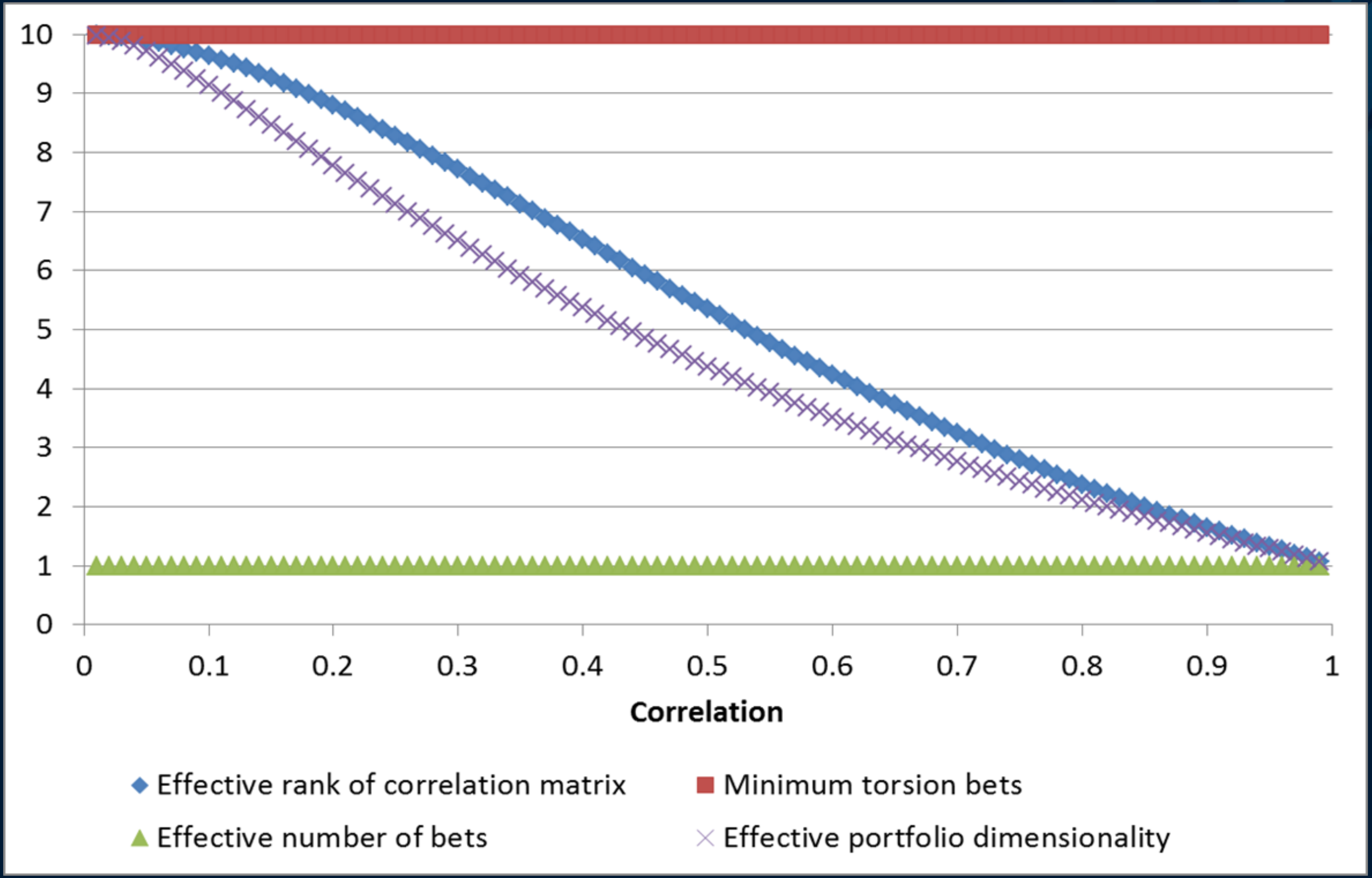
10 asset homogeneous correlation portfolio



Effective portfolio dimension (EPD)

- Portfolio diversification decomposition for ${\sigma_p}^2$ to determine $\exp(H(P_{EPD})$
- Portfolio construction inherently a problem of adding orthogonal risk
- Regression provides an alternative route

Homogeneous correlation market

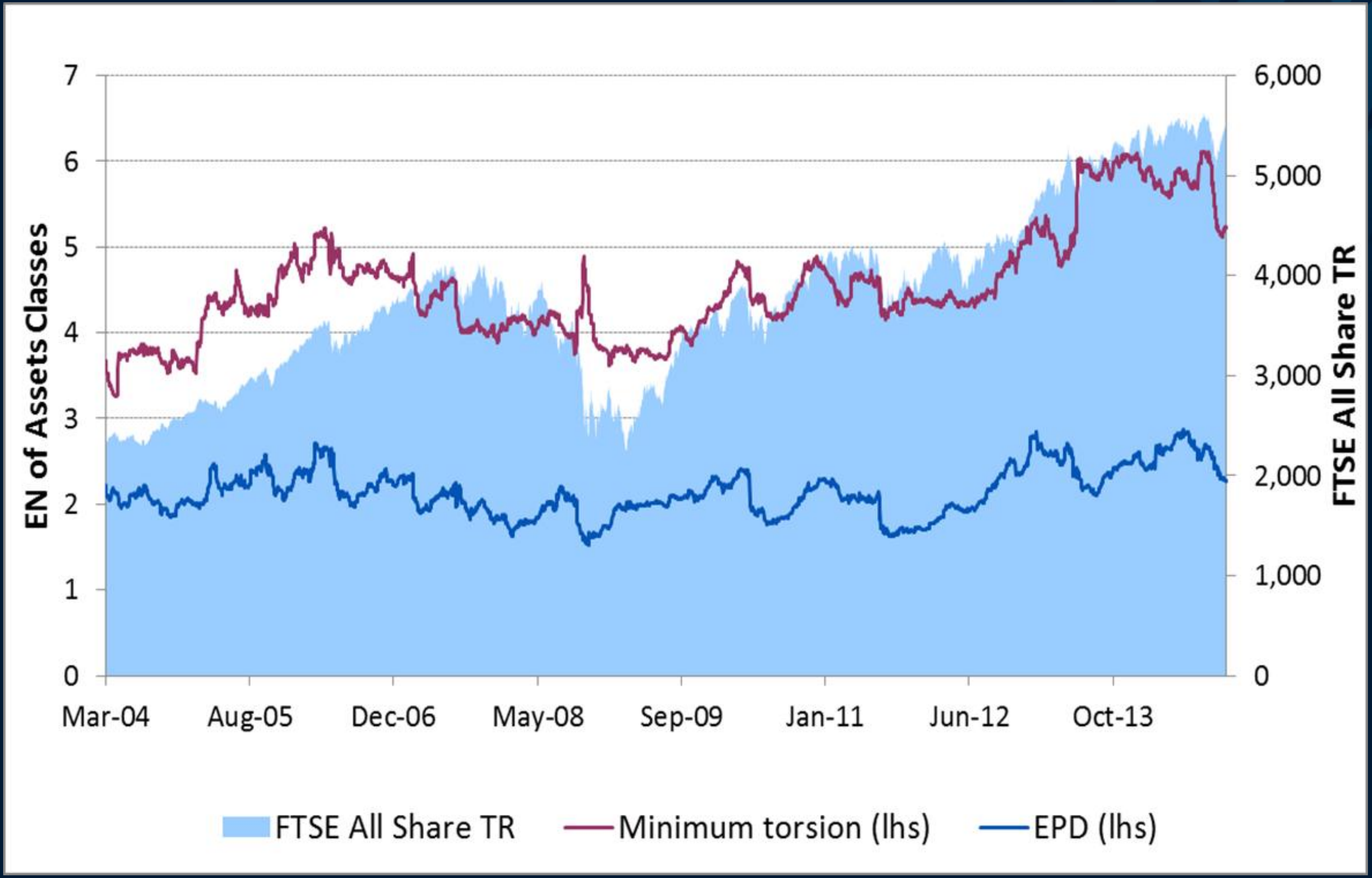


UK pension scheme example

Asset class	Index	Weighting*
Global equities	MSCI World	23.9%
UK equities	FTSE All-share	11.1%
ILGs	FTSE IL All-stock	18.1%
Gilts	FTSE All-stock	8.2%
GBP Non-gilts	ML GBP Non-gilt	17.8%
Hedge funds	HFR Global Hedge Funds	10.1%
Real estate	EPRA GBP REITs	4.6%
Cash	GBP 2m LIBOR	6.1%

* Purple Book 2014 – weighted average asset allocation

Minimum torsion vs EPD



Conclusions

- Diversification a hot topic with no standard measure
- Diversification inherently linked to market and portfolio health
- Dimensionality provides an intuitive approach
  - Interesting results for broader market and systemic risk
  - Useful links to PCA and variance explained
  - Some caution required applying to portfolios
- Recent work suggests a new direction for portfolio analysis

Thanks to the University of Edinburgh

Q & A