A Dynamic Contagion Process
with Applications to Finance & Insurance

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Outline

Background and Motivation

Default Contagion

One company's default triggers a series of other companies' default through their network of business and financial links.

Financial Crisis

Recently, the behavior of **default contagion** is more obvious during the current financial crisis, especially after the collapse of Lehman Brothers in September 2008.

Models in Literature *(Self Impact)*

A point process with its intensity process dependent on the point process itself could provide a more proper model to capture this contagion phenomenon.

- Jarrow and Yu (2001)
- Errais, Giesecke and Goldberg (2009)

Models in Literature *(External Impact)*

On the other hand, the default intensity could be impacted externally by multiple common factors, such as sector or market-wide events.

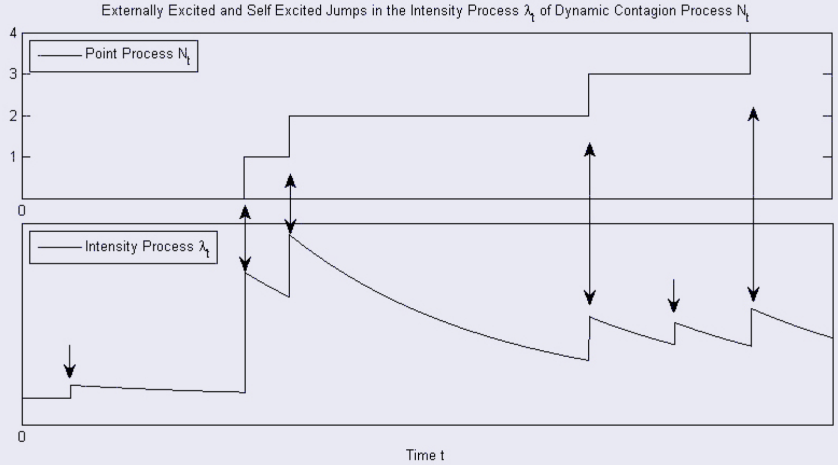
- Duffie and Gârleanu (2001)
- Longstaff and Rajan (2008)

Our Methodology *(Self + External Impact)*

We combine both of ideas by introducing the dynamic contagion process, a **new** point processes with both the **externally excited** and **self-excited** dependence structure.

- Hawkes (1971): Hawkes process (with exponential decay)
- Dassios and Jang (2003): Cox process with shot noise intensity
- Lando (1998): model the intensity of credit rating changing with Cox processes

Graphic Illustration *(Stochastic Intensity Representation)*



externally excited jumps $\left\{Y^{(1)}, T^{(1)}\right\}$ ($\downarrow$), self-excited jumps $\left\{Y^{(2)}, T^{(2)}\right\}$ ($\updownarrow$)

Mathematical Definition *(Stochastic Intensity Representation)*

The **dynamic contagion process** is a point process $N_t \equiv \left\{ T_k^{(2)} \right\}_{k=1,2,\ldots}$, with non-negative $\mathcal{F}_t-$stochastic intensity process λ_t following the piecewise deterministic dynamics with positive jumps,

$$\begin{aligned} \lambda_t &= a + (\lambda_0 - a)\, e^{-\delta t} \\ &+ \sum_{i \geq 1} Y_i^{(1)} e^{-\delta(t - T_i^{(1)})} \mathbb{I}\left\{ T_i^{(1)} \leq t \right\} + \sum_{k \geq 1} Y_k^{(2)} e^{-\delta(t - T_k^{(2)})} \mathbb{I}\left\{ T_k^{(2)} \leq t \right\}, \end{aligned}$$

where

- $\{\mathcal{F}_t\}_{t \geq 0}$ is a history of N_t, with respect to which $\{\lambda_t\}_{t \geq 0}$ is adapted,

Mathematical Definition *(Stochastic Intensity Representation)*

- $a \geq 0$ is the reversion level;
- $\lambda_0 > 0$ is the initial value of λ_t;
- $\delta > 0$ is the constant rate of exponential decay;
- $\left\{ Y_i^{(1)} \right\}_{i=1,2,\ldots}$ is a sequence of *i.i.d.* positive (externally excited) jumps with distribution $H(y)$, $y > 0$, at the corresponding random times $\left\{ T_i^{(1)} \right\}_{i=1,2,\ldots}$ following a homogeneous Poisson process M_t with constant intensity $\rho > 0$;
- $\left\{ Y_k^{(2)} \right\}_{k=1,2,\ldots}$ is a sequence of *i.i.d.* positive (self-excited) jumps with distribution $G(y)$, $y > 0$, at the corresponding random times $\left\{ T_k^{(2)} \right\}_{k=1,2,\ldots}$;
- The sequences $\left\{ Y_i^{(1)} \right\}_{i=1,2,\ldots}$, $\left\{ T_i^{(1)} \right\}_{i=1,2,\ldots}$ and $\left\{ Y_k^{(2)} \right\}_{k=1,2,\ldots}$ are assumed to be independent of each other.

Mathematical Definition *(Cluster Process Representation)*

The dynamic contagion process is a **cluster point process** $\mathbb{D}$ on $\mathbb{R}_+$: The number of points in the time interval $(0, t]$ is defined by $N_t = N_{\mathbb{D}(0,t]}$. The *cluster centers* of $\mathbb{D}$ are the particular points called *immigrants*, the other points are called *offspring*.They have the following structure:

- The *immigrants* are distributed according to a Cox process A with points $\{D_m\}_{m=1,2,\dots} \in (0,\infty)$ and shot noise stochastic intensity process

$$a + (\lambda_0 - a)\, e^{-\delta t} + \sum_{i \geq 1} Y_i^{(1)} e^{-\delta(t - T_i^{(1)})} \mathbb{I}\left\{ T_i^{(1)} \leq t \right\},$$

Mathematical Definition *(Cluster Process Representation)*

- Each *immigrant* D_m generates a *cluster* $C_m = C_{D_m}$, which is the random set formed by the points of *generations* $0, 1, 2, \ldots$ with the following branching structure:
  the *immigrant* D_m is said to be of *generation* 0. Given *generations* $0, 1, \ldots, j$ *in* C_m, each point $T^{(2)} \in C_m$ of *generation* j generates a Cox process on $(T^{(2)}, \infty)$ of *offspring* of *generation* $j+1$ with the stochastic intensity $Y^{(2)} e^{-\delta(\cdot - T^{(2)})}$ where $Y^{(2)}$ is a positive (self-excited) jump at time $T^{(2)}$ with distribution G, independent of the points of *generation* $0, 1, \ldots, j$.
- $\mathbb{D}$ consists of the union of all *clusters*, i.e.

$$\mathbb{D} = \bigcup_{m=1,2,\ldots} C_{D_m}.$$

Mathematical Definition *(Infinitesimal Generator Representation)*

The **infinitesimal generator** of the dynamic contagion process (λ_t, N_t, t) acting on $f(\lambda, n, t)$ within its domain $\Omega(\mathcal{A})$ is given by

$$\mathcal{A}f(\lambda,n,t) = \frac{\partial f}{\partial t} + \delta\,(a-\lambda)\,\frac{\partial f}{\partial \lambda} + \rho\left(\int_0^\infty f(\lambda+y,n,t)\mathrm{d}H(y) - f(\lambda,n,t)\right)$$
$$+\,\lambda\left(\int_0^\infty f(\lambda+y,n+1,t)\mathrm{d}G(y) - f(\lambda,n,t)\right) \qquad (1)$$

where $\Omega(\mathcal{A})$ is the domain for the generator $\mathcal{A}$ such that $f(\lambda, n, t)$ is differentiable with respect to λ, t for all λ, n and t, and

$$\left|\int_0^\infty f(\lambda+y,n,t)\mathrm{d}H(y) - f(\lambda,n,t)\right| < \infty$$

$$\left|\int_0^\infty f(\lambda+y,n+1,t)\mathrm{d}G(y) - f(\lambda,n,t)\right| < \infty$$

Joint Laplace Transform - Probability Generating Function of (λ_T, N_T)

Lemma

For the constants $0 \leq \theta \leq 1$ *and* $v \geq 0$*, we have the conditional joint Laplace transform - probability generating function for the intensity process* λ_t *and the point process* N_t*,*

$$\mathbb{E}\left[\theta^{(N_T - N_t)} \cdot e^{-v\lambda_T} \middle| \mathcal{F}_t\right] = e^{-\left(c(T)-c(t)\right)} e^{-B(t)\lambda_t} \tag{2}$$

where $B(t)$ *is determined by the non-linear ODE*

$$-B'(t) + \delta B(t) + \theta \cdot \hat{g}\big(B(t)\big) - 1 = 0 \tag{3}$$

with boundary condition $B(T) = v$*. Then,* $c(T) - c(t)$ *is determined by*

$$c(T) - c(t) = a\delta \int_t^T B(s)\mathrm{d}s + \rho \int_t^T \left[1 - \hat{h}\big(B(s)\big)\right] \mathrm{d}s \tag{4}$$

Theorem

The conditional Laplace transform of λ_T given λ_0 at time $t = 0$, under condition $\delta > \mu_{1_G}$, is given by

$$\mathbb{E}\left[e^{-v\lambda_T}\middle|\lambda_0\right] = \exp\left(-\int_{\mathcal{G}_{v,1}^{-1}(T)}^{v} \frac{a\delta u + \rho[1-\hat{h}(u)]}{\delta u + \hat{g}(u) - 1}\mathrm{d}u\right) \times e^{-\mathcal{G}_{v,1}^{-1}(T)\cdot\lambda_0} \tag{5}$$

where the well defined (strictly decreasing) function

$$\mathcal{G}_{v,1}(L) =: \int_{L}^{v} \frac{\mathrm{d}u}{\delta u + \hat{g}(u) - 1}$$

$$\mu_{1_G} =: \int_0^\infty y\mathrm{d}G(y); \quad \hat{g}(u) =: \int_0^\infty e^{-uy}\mathrm{d}G(y); \quad \hat{h}(u) =: \int_0^\infty e^{-uy}\mathrm{d}H(y)$$

Stationary Laplace Transform of λ_T

Let $T \to \infty$, then $\mathcal{G}_{v,1}^{-1}(T) \to 0$, we have

Theorem

The Laplace transform of ***asymptotic distribution*** *of λ_T, under condition $\delta > \mu_{1_G}$, is given by*

$$\lim_{T\to\infty} \mathbb{E}\left[e^{-v\lambda_T}\middle|\lambda_0\right] = \exp\left(-\int_0^v \frac{a\delta u + \rho[1-\hat{h}(u)]}{\delta u + \hat{g}(u) - 1}\mathrm{d}u\right) \tag{6}$$

and this is also the Laplace transform of ***stationary distribution*** *of process $\{\lambda_t\}_{t\geq 0}$.*

Example: Jumps with Exponential Distributions

Example

Externally excited and self-excited jumps follow exponential distributions with parameters α and β, explicitly,

$$\hat{h}(u) = \frac{\alpha}{\alpha + u}; \quad \hat{g}(u) = \frac{\beta}{\beta + u} \tag{7}$$

Example: Jumps with Exponential Distributions

Example

By identifying from Laplace transform, λ_T can be decomposed into two independent random variables plus constant a,

$$\lambda_T \stackrel{\mathcal{D}}{=} \begin{cases} a + \tilde{\Gamma}_1 + \tilde{\Gamma}_2 & \text{for } \alpha \geq \beta \\ a + \tilde{\Gamma}_3 + \tilde{B} & \text{for } \alpha < \beta \text{ and } \alpha \neq \beta - \frac{1}{\delta} \\ a + \tilde{\Gamma}_4 + \tilde{P} & \text{for } \alpha = \beta - \frac{1}{\delta} \end{cases}$$

where

$$\tilde{\Gamma}_1 \sim \text{Gamma}\left(\frac{1}{\delta}\left(a + \frac{\rho}{\delta(\alpha-\beta)+1}\right), \frac{\delta\beta-1}{\delta}\right);$$

$$\tilde{\Gamma}_2 \sim \text{Gamma}\left(\frac{\rho(\alpha-\beta)}{\delta(\alpha-\beta)+1}, \alpha\right);$$

$$\tilde{\Gamma}_3 \sim \text{Gamma}\left(\frac{a+\rho}{\delta}, \frac{\delta\beta-1}{\delta}\right); \quad \tilde{\Gamma}_4 \sim \text{Gamma}\left(\frac{a+\rho}{\delta}, \alpha\right);$$

Example

$$\tilde{B} \stackrel{\mathcal{D}}{=} \sum_{i=1}^{N_1} X_i^{(1)}, \quad N_1 \sim \mathrm{NegBin}\left(\frac{\rho}{\delta}\frac{\beta-\alpha}{\gamma_1-\gamma_2}, \frac{\gamma_2}{\gamma_1}\right), X_i^{(1)} \sim \mathrm{Exp}(\gamma_1);$$

$$\tilde{P} \stackrel{\mathcal{D}}{=} \sum_{i=1}^{N_2} X_i^{(2)}, \quad N_2 \sim \mathrm{Poisson}\left(\frac{\rho}{\delta^2\alpha}\right), X_i^{(2)} \sim \mathrm{Exp}(\alpha)$$

and $\gamma_1 = \max\left\{\alpha, \frac{\delta\beta-1}{\delta}\right\}$, $\gamma_2 = \min\left\{\alpha, \frac{\delta\beta-1}{\delta}\right\}$; $\tilde{B}$ follows a compound negative binomial distribution with underlying exponential jumps; $\tilde{P}$ follows a compound Poisson distribution with underlying exponential jumps.

Example

Special cases:

- Dassios and Jang (2003): $\beta = \infty$

$$\lambda_T \stackrel{\mathcal{D}}{=} a + \tilde{\Gamma}_5, \qquad \tilde{\Gamma}_5 \sim \text{Gamma}\left(\frac{\rho}{\delta}, \alpha\right)$$

- Hawkes process (1971): $\alpha = \infty$, or $\rho = 0$

$$\lambda_T \stackrel{\mathcal{D}}{=} a + \tilde{\Gamma}_6, \qquad \tilde{\Gamma}_6 \sim \text{Gamma}\left(\frac{a}{\delta}, \frac{\delta\beta - 1}{\delta}\right)$$

Probability Generating Function of N_T

Theorem

The conditional probability generating function of N_T given λ_0 and $N_0 = 0$ at time $t = 0$, under condition $\delta > \mu_{1_G}$, is given by

$$\mathbb{E}\left[\theta^{N_T} \middle| \lambda_0\right] = \exp\left(-\int_0^{\mathcal{G}_{0,\theta}^{-1}(T)} \frac{a\delta u + \rho[1-\hat{h}(u)]}{1-\delta u-\theta\cdot\hat{g}(u)}\mathrm{d}u\right) \times e^{-\mathcal{G}_{0,\theta}^{-1}(T)\cdot\lambda_0}$$

where the well defined (strictly increasing) function

$$\mathcal{G}_{0,\theta}(L) =: \int_0^L \frac{\mathrm{d}u}{1-\delta u-\theta\cdot\hat{g}(u)} \qquad 0 \le \theta < 1$$

- default is caused by a series of "**bad events**" released from the underlying company;
- each bad event can result to default with probability d;
- d measures the capability to avoid bankruptcy (e.g. credit ratings);
- the conditional **survival probability** at time T is

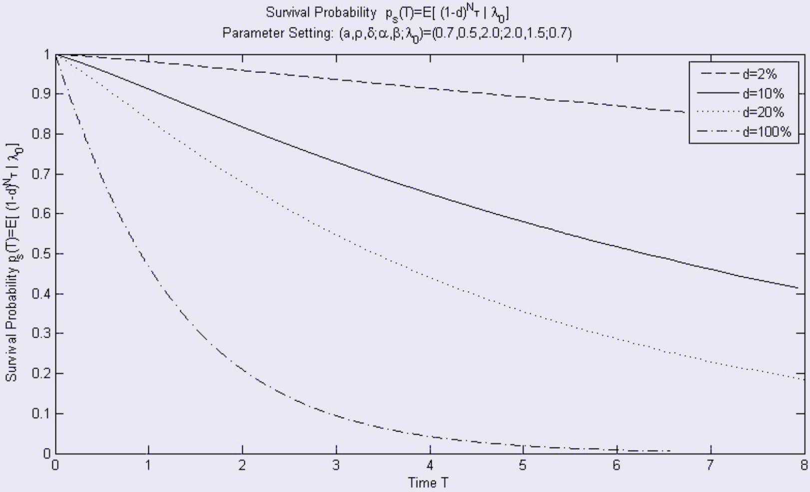
$$p_s(T) = \mathbb{E}\left[(1-d)^{N_T} \middle| \lambda_0\right]$$

- set the parameters $(a, \rho, \delta; \alpha, \beta; \lambda_0) = (0.7, 0.5, 2.0; 2.0, 1.5; 0.7)$.

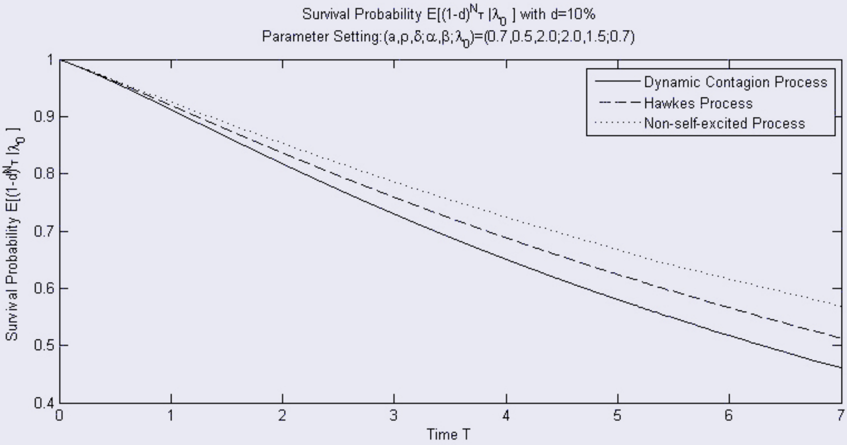
Table: Survival Probability $p_s(T)$

Time T	1	2	3	4	5	6
$d = 2\%$	98.15%	95.92%	93.65%	91.40%	89.21%	87.06%
$d = 10\%$	91.26%	81.78%	72.99%	65.07%	58.01%	51.70%
$d = 20\%$	83.66%	67.91%	54.78%	44.13%	35.54%	28.63%
$d = 100\%$	46.73%	21.10%	9.48%	4.26%	1.92%	0.86%

Survival Probability



Comparison for Survival Probabilities under Three Processes



Surplus Process

The claim arrivals are modelled by *dynamic contagion process* (N_t, λ_t), i.e. for surplus process X_t,

$$X_t = X_0 + ct - \sum_{i=1}^{N_t} Z_i \qquad (t \geq 0) \tag{8}$$

where

- $X_0 = x \geq 0$ is the initial reserve at time $t = 0$;
- $c > 0$ is the constant rate of premium payment per time unit;
- N_t is the dynamic contagion process ($N_0 = 0$) counting the number of claims arriving in the time interval $(0, t]$, with intensity process λ_t, given $\lambda_0 = \lambda > 0$;
- $\{Z_i\}_{i=1,2,...}$ is a sequence of *i.i.d.* positive random variables (claim sizes) with distribution $Z(z), z > 0$, and independent of N_t.

Ruin Probability

The *stopping time* τ^* is the first time of ruin for X_t,

$$\tau^* =: \begin{cases} \inf\left\{t > 0 \middle| X_t \leq 0\right\} & \\ \inf\{\varnothing\} = \infty & \textit{if } X_t > 0 \textit{ for all } t. \end{cases}$$

We are interested in the *ruin probability* in finite time,

$$\phi(x, \lambda, t) =: P\left\{\tau^* < t \middle| X_0 = x, \lambda_0 = \lambda\right\};$$

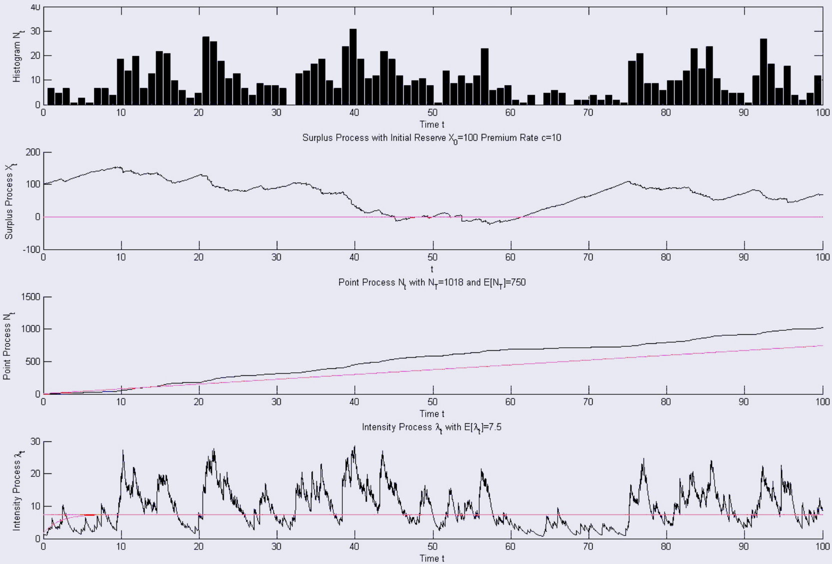
particularly, the *ultimate ruin probability* in infinite time,

$$\phi(x, \lambda) =: P\left\{\tau^* < \infty \middle| X_0 = x, \lambda_0 = \lambda\right\};$$

and also when the intensity process λ_t is stationary,

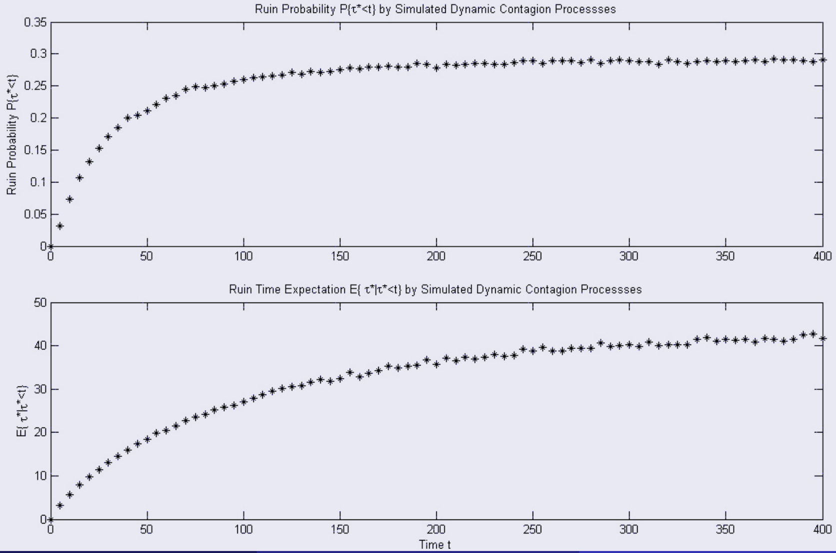
$$\phi(x) =: P\left\{\tau^* < \infty \middle| X_0 = x, \lambda_0 = \lambda \sim \Pi\right\}.$$

One Simulated Sample Path (with Ruin)



Ruin by Simulation

$P\left\{\tau^* < t \middle| X_0 = x, \lambda_0 = \lambda\right\}$ and $\mathbb{E}\left[\tau^* < t \middle| X_0 = x, \lambda_0 = \lambda\right]$



Theorem

If the claim arrivals of the surplus process X_t is driven by *dynamic contagion process* (N_t, λ_t), under condition $\delta > \mu_{1_G}$, then, we have **net profit condition**

$$c > \frac{\mu_{1_H}\rho + a\delta}{\delta - \mu_{1_G}} \cdot \mu_{1_Z} \qquad \left(\delta > \mu_{1_G}\right), \tag{9}$$

where

$$\mu_{1_Z} =: \int_0^\infty z \mathrm{d}Z(z).$$

If net profit condition holds, then ruin in infinite is not certain, i.e.

$$\lim_{t\to\infty} X_t = \infty \quad \text{or,} \quad P\{\tau^* < \infty\} < 1$$

Theorem

Under $\delta > \mu_{1_G}$ and net profit condition,

$$e^{-v_r X_t} e^{\eta_r \lambda_t} e^{-rt} \quad (r \geq 0) \tag{10}$$

is a **martingale**, where constants $r \geq 0$, v_r and η_r satisfy a *generalized Lundberg's Fundamental Equation*

$$\begin{cases} \delta\xi_r + \hat{z}(-v_r)\hat{g}(-\eta_r) - 1 = 0 & (.1) \\ \rho\left(\hat{h}(-\eta_r) - 1\right) - r + a\delta\eta_r - cv_r = 0 & (.2) \end{cases} \tag{11}$$

where

$$\hat{z}(u) =: \int_0^\infty e^{-uz} \mathrm{d}z(z).$$

Theorem

- For $0 \leq r < \bar{r}$, we have unique solution $\left(v_r^+ > 0, \eta_r^+ > 0\right)$;
- for $r = 0$, unique solution $\left(v_0^+ > 0, \eta_0^+ > 0\right)$,
  where

$$\bar{r} = \rho\left(\hat{h}(-\bar{\eta}) - 1\right) + a\delta\bar{\eta}, \tag{12}$$

and the constant $\bar{\eta}$ is the unique positive solution to

$$1 + \delta\eta_r = \hat{g}(-\eta_r) \qquad \left(\delta > \mu_{1_G}\right). \tag{13}$$

Theorem

We use the unique martingale $e^{-v_0^+ X_t} e^{\eta_0^+ \lambda_t}$ to define an **equivalent probability measure** $\widetilde{\mathbb{P}}$ via the *Radon-Nikodym derivative*

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} =: e^{-v_0^+(X_t - x)} e^{\eta_0^+(\lambda_t - \lambda)} \tag{14}$$

with $\mathbb{P} \to \widetilde{\mathbb{P}}$ parameter transformation by

- $c \to c$, $\delta \to \delta$,
- $a \nearrow \left(1 + \delta\eta_0^+\right) a$,
- $\rho \nearrow \hat{h}(-\eta_0^+)\rho$,
- $Z(z) \to \widetilde{Z}(z)$,
- $g(u) \to \dfrac{\widetilde{g}\left(\frac{u}{1+\delta\eta_0^+}\right)}{1+\delta\eta_0^+}$, $h(u) \to \dfrac{\widetilde{h}\left(\frac{u}{1+\delta\eta_0^+}\right)}{1+\delta\eta_0^+}$.

Net Profit Condition under $\widetilde{\mathbb{P}}$

Theorem

If the net profit condition and the stationarity condition both hold under original measure $\mathbb{P}$, i.e.

$$c > \frac{\mu_{1_H}\rho + a\delta}{\delta - \mu_{1_G}} \cdot \mu_{1_Z}, \quad \delta > \mu_{1_G}, \tag{15}$$

and the stationarity condition also holds under new measure $\widetilde{\mathbb{P}}$, i.e. $\widetilde{\delta} > \mu_{1_{\widetilde{G}}}$, then, under measure $\widetilde{\mathbb{P}}$, we have

$$\frac{\mu_{1_{\widetilde{H}}}\widetilde{\rho} + \widetilde{a}\widetilde{\delta}}{\widetilde{\delta} - \mu_{1_{\widetilde{G}}}} \cdot \mu_{1_{\widetilde{Z}}} > \widetilde{c}, \tag{16}$$

and ruin becomes certain (*almost surely*), i.e.

$$\widetilde{\mathbb{P}}\{\tau^* < \infty\} =: \lim_{t\to\infty} \widetilde{\mathbb{P}}\{\tau^* \le t\} = 1. \tag{17}$$

Theorem

Assume the net profit condition holds under $\mathbb{P}$, and the stationarity condition holds under $\mathbb{P}$ and $\widetilde{\mathbb{P}}$, then

$$\begin{aligned} & P\left\{\tau^* < \infty \middle| X_0 = x, \lambda_0 = \lambda\right\} \\ = \quad & e^{-v_0^+ x} e^{m\widetilde{\lambda}} \cdot \widetilde{\mathbb{E}}\left[\Psi\left(X_{\tau_-^*}\right)\frac{e^{-m\widetilde{\lambda}_{\tau_-^*}}}{\hat{g}(-\eta_0^+)}\middle| X_0 = x, \widetilde{\lambda}_0 = \widetilde{\lambda}\right] \end{aligned} \tag{18}$$

where $m = \frac{\eta_0^+}{\delta\eta_0^+ + 1}$, $\widetilde{\lambda} = (1 + \delta\eta_0^+)\lambda$,

$$\Psi(x) =: \frac{\bar{Z}(x)e^{v_0^+ x}}{\int_x^\infty e^{v_0^+ z}\mathrm{d}Z(z)}. \tag{19}$$

Generalization: Discretised Dynamic Contagion Process

The **discretised dynamic contagion process** $\{(N_t, M_t)\}_{t\geq 0}$ is a point process on $\mathbb{R}_+$ such that

$$
\begin{aligned}
& P\left\{M_{t+\Delta t}-M_t=k, N_{t+\Delta t}-N_t=0 \middle| M_t, N_t\right\} \\
= \ & \rho p_k \Delta t+o(\Delta t), \quad k=1,2..., \\
& P\left\{M_{t+\Delta t}-M_t=k-1, N_{t+\Delta t}-N_t=1 \middle| M_t, N_t\right\} \\
= \ & \delta M_t q_k \Delta t+o(\Delta t), \quad k=0,1..., \\
& P\left\{M_{t+\Delta t}-M_t=0, N_{t+\Delta t}-N_t=0 \middle| M_t, N_t\right\} \\
= \ & 1-\big(\rho(1-p_0)+\delta M_t\big)\Delta t+o(\Delta t), \\
& P\left\{\text{Others} \middle| M_t, N_t\right\}=o(\Delta t),
\end{aligned}
$$

where

- $\delta, \rho > 0$ are constants;
- independent jumps K_P and joint jumps K_Q are two types of jumps in process M_t, with probabilities given respectively by

$$
p_k =: P\{K_P=k\}, \quad q_k =: P\{K_Q=k\}, \quad k=0,1....
$$

Discretised Dynamic Contagion Process

We could use it to model the interim payments (claims) in insurance, if we assume

- N_t is the number of cumulative settled claims within $[0, t]$;
- M_t is denoted as the number of cumulative unsettled claims $[0, t]$;
- the arrival of clusters of claims follow a Poisson process of rate ρ;
- there are random number K_P of claims with probability p_k occurring simultaneously at each cluster;
- each of the claims will be settled with exponential delay of rate δ;
- at each of the settlement times, only one claim can be settled, however, a random number K_Q of new claims with probability q_k could be revealed and need further settlement.

Theorem

The discretised dynamic contagion process is a zero-reversion dynamic contagion process, if

$$\begin{aligned} K_P &\sim \textit{Mixed–Poisson}\left(\frac{Y}{\delta}\middle| Y \sim H\right), \\ K_Q &\sim \textit{Mixed–Poisson}\left(\frac{Y}{\delta}\middle| Y \sim G\right), \end{aligned}$$

i.e.

$$p_k = \int_0^\infty \frac{e^{-\frac{y}{\delta}}}{k!}\left(\frac{y}{\delta}\right)^k \mathrm{d}H(y), \quad q_k = \int_0^\infty \frac{e^{-\frac{y}{\delta}}}{k!}\left(\frac{y}{\delta}\right)^k \mathrm{d}G(y).$$

A Special Case: A Risk Model with Delayed Claims

Consider a surplus process $\{X_t\}_{t\geq 0}$,

$$X_t = x + ct - \sum_{i=1}^{N_t} Z_i, \quad t \geq 0,$$

where

- $x = X_0 \geq 0$ is the initial reserve at time $t = 0$;
- $c > 0$ is the constant rate of premium payment per time unit;
- N_t is the number of cumulative settled claims within $[0, t]$;
- $\{Z_i\}_{i=1,2,\ldots}$ is a sequence of i.i.d. r.v. with the cumulative distribution $Z(z), z > 0$, the mean and tail of Z are denoted respectively by

$$\mu_{1_Z} = \int_0^\infty z \mathrm{d}Z(z), \quad \overline{Z}(x) = \int_x^\infty \mathrm{d}Z(s).$$

A Risk Model with Delayed Claims

- Assume the arrival of claims follows a Poisson process of rate ρ, and each of the claims will be settled with a random delay.
- Loss only occurs when claims are being settled.
- M_t is denoted as the number of cumulative unsettled claims within the time interval $[0, t]$ and assume the initial number $M_0 = 0$.
- $\{T_k\}_{k=1,2,\ldots}$, $\{L_k\}_{k=1,2,\ldots}$ and $\{T_k + L_k\}_{k=1,2,\ldots}$ are denoted as the (random) times of claim arrival, delay and settlement, respectively, and hence,

$$M_t = \sum_k \Big(\mathbb{I}\{T_k \le t\} - \mathbb{I}\{T_k + L_k \le t\} \Big), \quad N_t = \sum_k \mathbb{I}\{T_k + L_k \le t\}.$$

By Mirasol (1963), a delayed (or displaced) Poisson process is still a (non-homogeneous) Poisson process.
It is a special case of discretised dynamic contagion process if L is exponentially distributed.

The ruin (stopping) time after time $t \geq 0$ is defined by

$$\tau_t^* =: \begin{cases} \inf \{s : s > t, X_s \leq 0\}, & \\ \inf\{\varnothing\} = \infty, & \text{if } X_s > 0 \text{ for all } s; \end{cases}$$

in particular, $\tau_t^* = \infty$ means ruin does not occur. We are interested in the ultimate ruin probability at time t, i.e.

$$\psi(x, t) =: P\left\{\tau_t^* < \infty \middle| X_t = x\right\},$$

or, the ultimate non-ruin probability at time t, i.e.

$$\phi(x, t) =: 1 - \psi(x, t).$$

Lemma

Assume $c > \rho\mu_{1_Z}$ and $L \sim \mathrm{Exp}(\delta)$, we have a series of modified Lundberg fundamental equations

$$cw - \rho\left[1 - \hat{z}(w)\right] - \delta j = 0, \quad j = 0, 1, ...; \tag{20}$$

- *for $j = 0$, (20) has solution zero and a unique negative solution (denoted by $W_0^+ = 0$ and $W_0^- < 0$);*
- *for $j = 1, 2, ...$, (20) has unique positive and negative solutions (denoted by $W_j^+ > 0$ and $W_j^- < 0$).*

Denote the (modified) adjustment coefficients by $R_j =: -W_j^-, j = 0, 1, ...$; note that, $0 < R_0 < R_1 < R_2 < ... < R_\infty$, where $R_\infty =: \inf\left\{R \middle| \hat{z}(-R) = \infty\right\}$.

Theorem

Assume $c > \rho\mu_{1_Z}$ and the first, second moments of L exist, we have the asymptotics of ruin probability

$$\psi(x,t) \sim e^{-cR_0\int_t^\infty \overline{L}(s)\mathrm{d}s}\frac{c-\rho\mu_{1_Z}}{\rho\int_0^\infty ze^{R_0 z}\mathrm{d}Z(z)-c}e^{-R_0 x}+o\left(e^{-R_0 x}\right), x\to\infty,$$

where $\overline{L}(t) =: 1 - L(t)$.

Theorem

Assume $c > \rho\mu_{1_Z}$ and $L \sim \mathrm{Exp}(\delta)$, we have the Laplace transform of non-ruin probability

$$\hat{\phi}(w,t) = \\ = e^{\vartheta e^{-\delta t}[1-\hat{z}(w)]}\left(\frac{c-\rho\mu_{1_Z}}{cw-\rho\left[1-\hat{z}(w)\right]} + c\sum_{j=1}^{\infty} e^{-j\delta t}\frac{\sum_{\ell=0}^{j} r_\ell \frac{[\vartheta\hat{z}(w)]^{j-\ell}}{(j-\ell)!}}{cw-\rho\left[1-\hat{z}(w)\right]-\delta j}\right),$$

where $\vartheta = \frac{\rho}{\delta}$,

$$r_0 = 1 - \frac{\rho}{c}\mu_{1_Z}, \quad r_\ell = -\sum_{i=0}^{\ell-1}\frac{\left[\vartheta\hat{z}(W_\ell^+)\right]^{\ell-i}}{(\ell-i)!}r_i, \quad \ell = 1,2,\ldots.$$

Theorem

Assume $c > \rho\mu_{1_Z}$ and $L \sim \mathrm{Exp}(\delta)$, we have the Laplace transform of the non-ruin probability

$$\hat{\phi}(w,t) = \sum_{j=0}^{\infty} e^{-j\delta t}\hat{\phi}_j(w),$$

where $\left\{\hat{\phi}_j(w)\right\}_{j=0,1,\ldots}$ follow the recurrence

$$\hat{\phi}_j(w) = \rho\frac{\left[1-\hat{z}(W_j^+)\right]\hat{\phi}_{j-1}(W_j^+) - [1-\hat{z}(w)]\,\hat{\phi}_{j-1}(w)}{cw-\rho\,[1-\hat{z}(w)]-\delta j}, \quad j=1,2,\ldots,$$

$$\hat{\phi}_0(w) = \frac{c\left(1-\frac{\rho}{c}\mu_{1_Z}\right)}{cw-\rho\,[1-\hat{z}(w)]}.$$

Theorem

Assume $c > \rho\mu_{1_Z}$, $L \sim \mathrm{Exp}(\delta)$, the asymptotics of ruin probability is

$$\psi(x,t) \sim \sum_{j=0}^{\infty} \kappa_j(t) e^{-R_j x}, \quad x \to \infty,$$

$$\kappa_0(t) =: e^{-\frac{cR_0}{\rho}\vartheta e^{-\delta t}} \frac{c - \rho\mu_{1_Z}}{\rho \int_0^\infty z e^{R_0 z} \mathrm{d}Z(z) - c},$$

$$\kappa_j(t) =: e^{-j\delta t} \frac{c e^{\vartheta e^{-\delta t}[1 - \hat{z}(-R_j)]}}{\rho \int_0^\infty z e^{R_j z} \mathrm{d}Z(z) - c} \sum_{\ell=0}^{j} r_\ell \frac{[\vartheta \hat{z}(-R_j)]^{j-\ell}}{(j-\ell)!}, \quad j = 1, 2, \ldots.$$

If Z follows an exponential distribution, we have

$$\psi(x,t) = \sum \kappa_j(t) e^{-R_j x}.$$

Reference

- Dassios, A., Zhao, H. (2011). A Dynamic Contagion Process. *Advances in Applied Probability* **43(3)** 814-846.
- Dassios, A., Zhao, H. (2012). Ruin by Dynamic Contagion Claims. *Insurance: Mathematics and Economics*. **51(1)** 93-106.
- Dassios, A., Zhao, H. (2011). A Risk Model with Delayed Claims. To appear in *Journal of Applied Probability*.
- Dassios, A., Zhao, H. (2012). A Markov Chain Model for Contagion. Submitted.
- Dassios, A., Zhao, H. (2012). A Dynamic Contagion Process with Diffusion. Working paper.