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Probability Weighted Moments

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28 November 2014

Introduction

- If I asked you to summarise a data set, or fit a distribution …
- You'd probably calculate the mean and standard deviation
- … followed by skewness and kurtosis.

- But there is an alternative, popular with hydrologists
- They are called L-moments, or Probability Weighted Moments
- This presentation considers whether L-moments could have a role in actuarial work.

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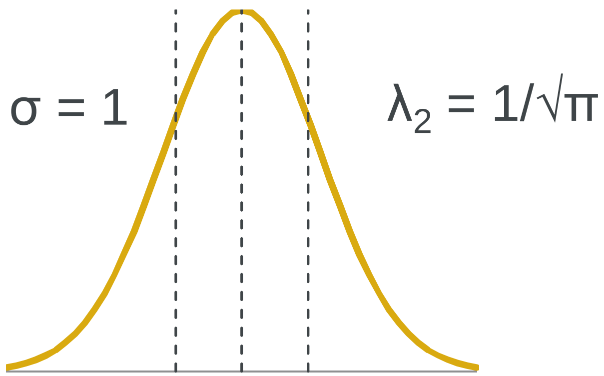
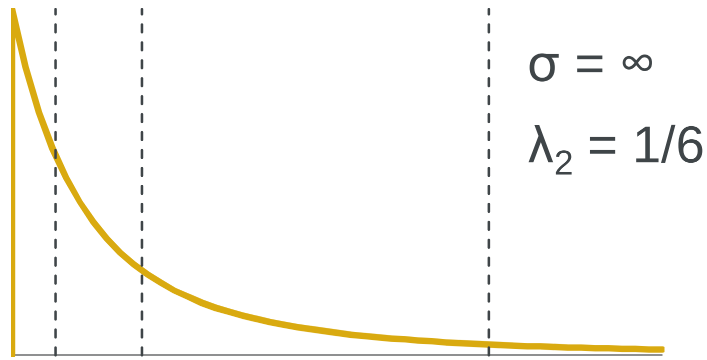
Standard Deviation and L-Scale

- Stdev $\sigma = \{\mathbf{E}(X-\mu)^2\}^{1/2}$
- Where $\mu = \mathbf{E}(X)$
- Or, $\sigma = \{\mathbf{E}(X_1-X_2)^2/2\}^{1/2}$
- For X_1, X_2 independent

- L-scale $\lambda_2 = \mathbf{E}|X_1-X_2|/2$
- For X_1, X_2 independent
- Or, $\lambda_2 = [\mathbf{E}\max\{X_1,X_2\}-\mathbf{E}\min\{X_1,X_2\}]/2$

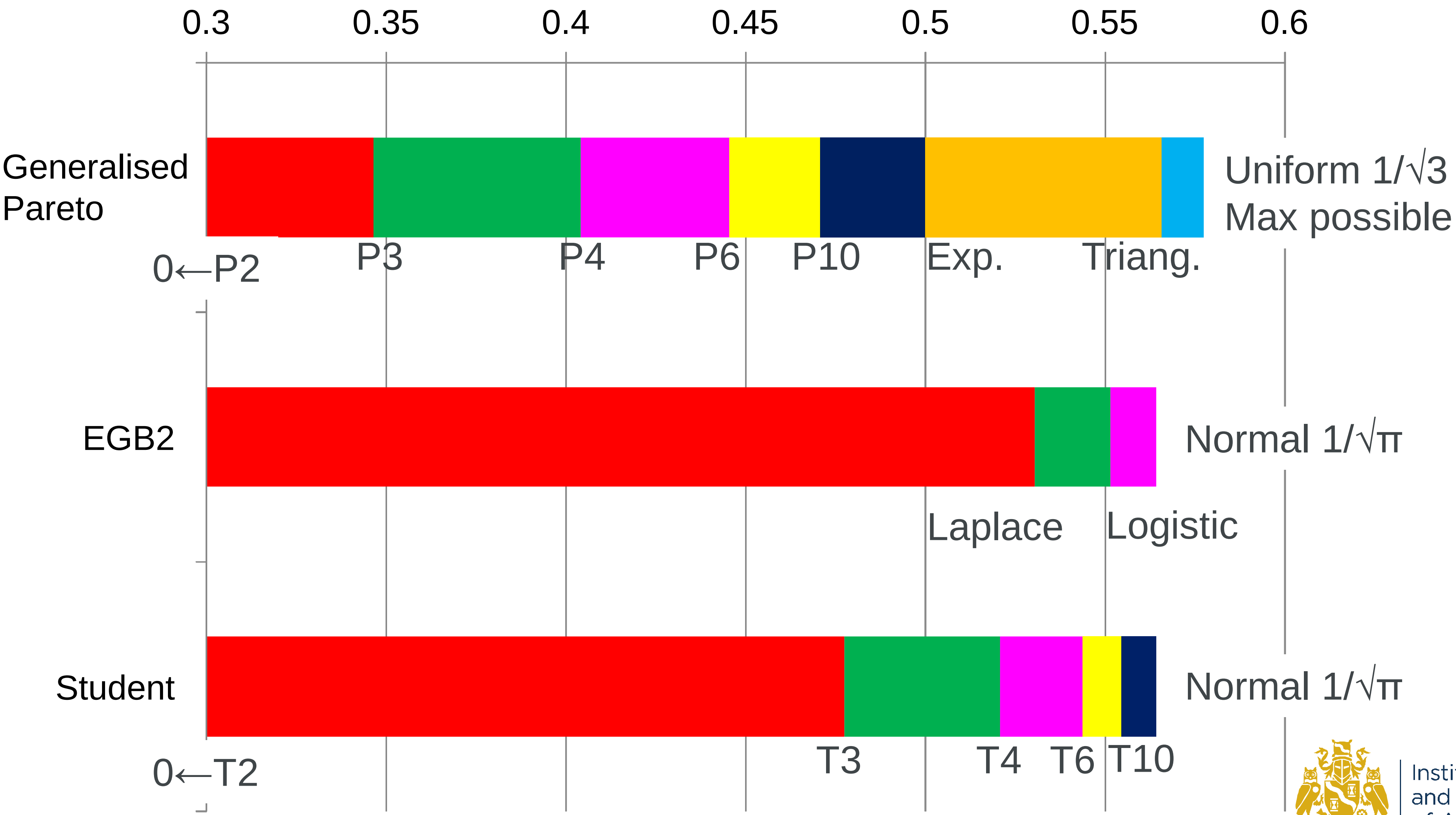


Standard normal density



Pareto density (α=2)

Does the choice make any difference? Expressing λ_2 as a multiple of σ



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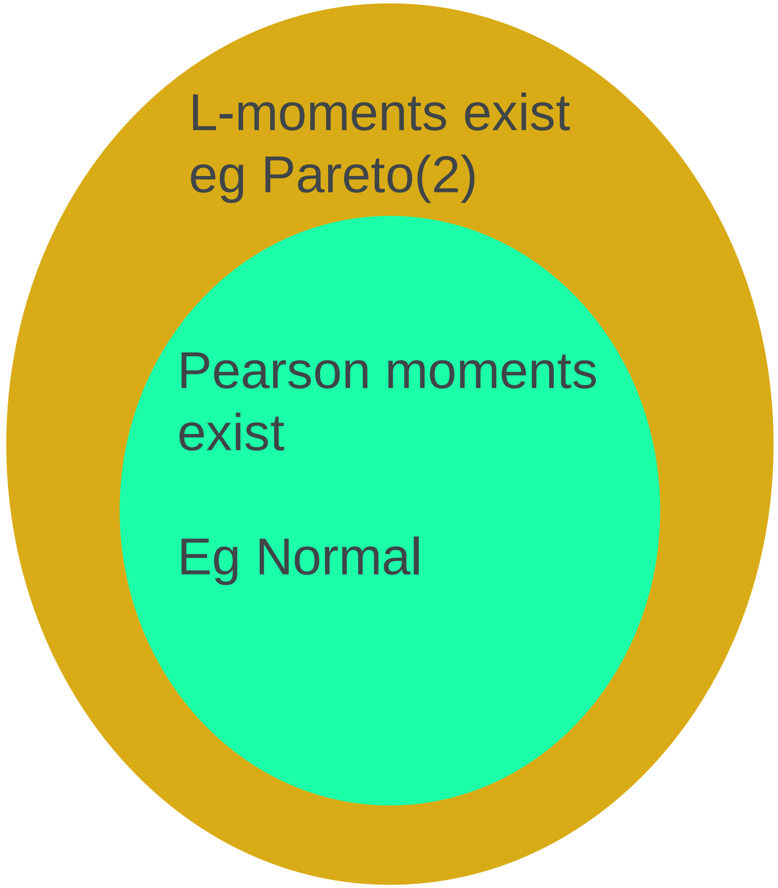
Sampling Behaviour

- Hydrologists prefer L-moments because of nice sampling behaviour
  - Less sensitive to outliers
  - Lower sampling variability
  - Fast convergence to asymptotic normality
- Same rationale might apply to actuarial work.



Model Mis-Specification Error

- Given sufficient data, we might be able to the L-scale or the standard deviation reasonably accurately
- But we still face estimation error if we plug that estimate into the wrong distribution.
- Chebyshev-style inequalities suggest things can go very badly wrong, but the situation is better if we focus on nice bell-shaped distributions.
- In the next two slides we consider an ambiguity set of models containing {Weibull, Normal, logistic, Laplace, T3, T4, T6 and T10}.
- We ask how wrong we could be if we try to calculate a 1-in-200 event, with the right input parameter but the wrong model.

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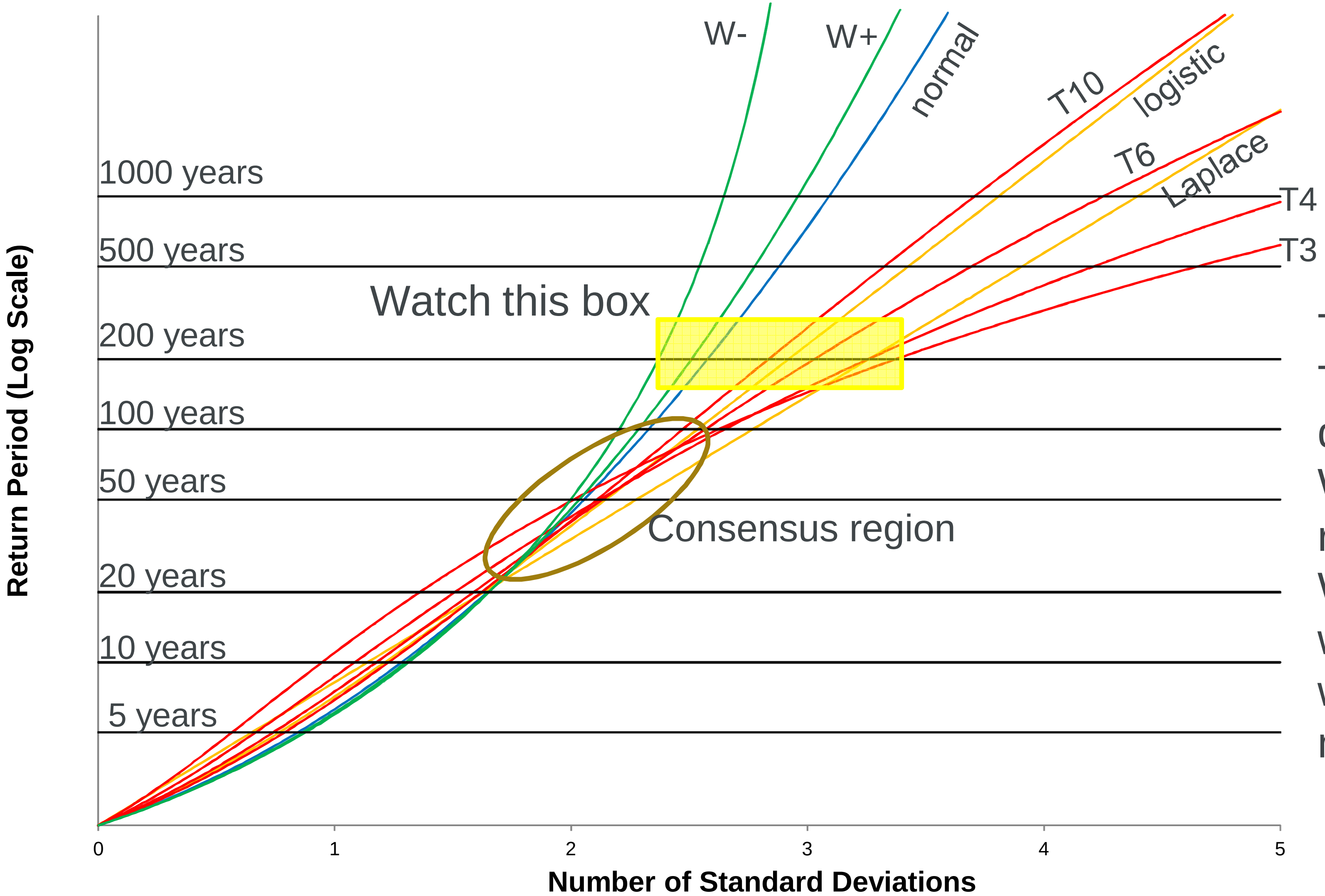
Impact of Model Uncertainty
Using σ as a Proxy for Value-at-Risk



Tv refer to Student's T distributions with v degrees of freedom. W+ and W- are the right and left tails of a Weibull distribution with k = 3.43954 where mean = median.

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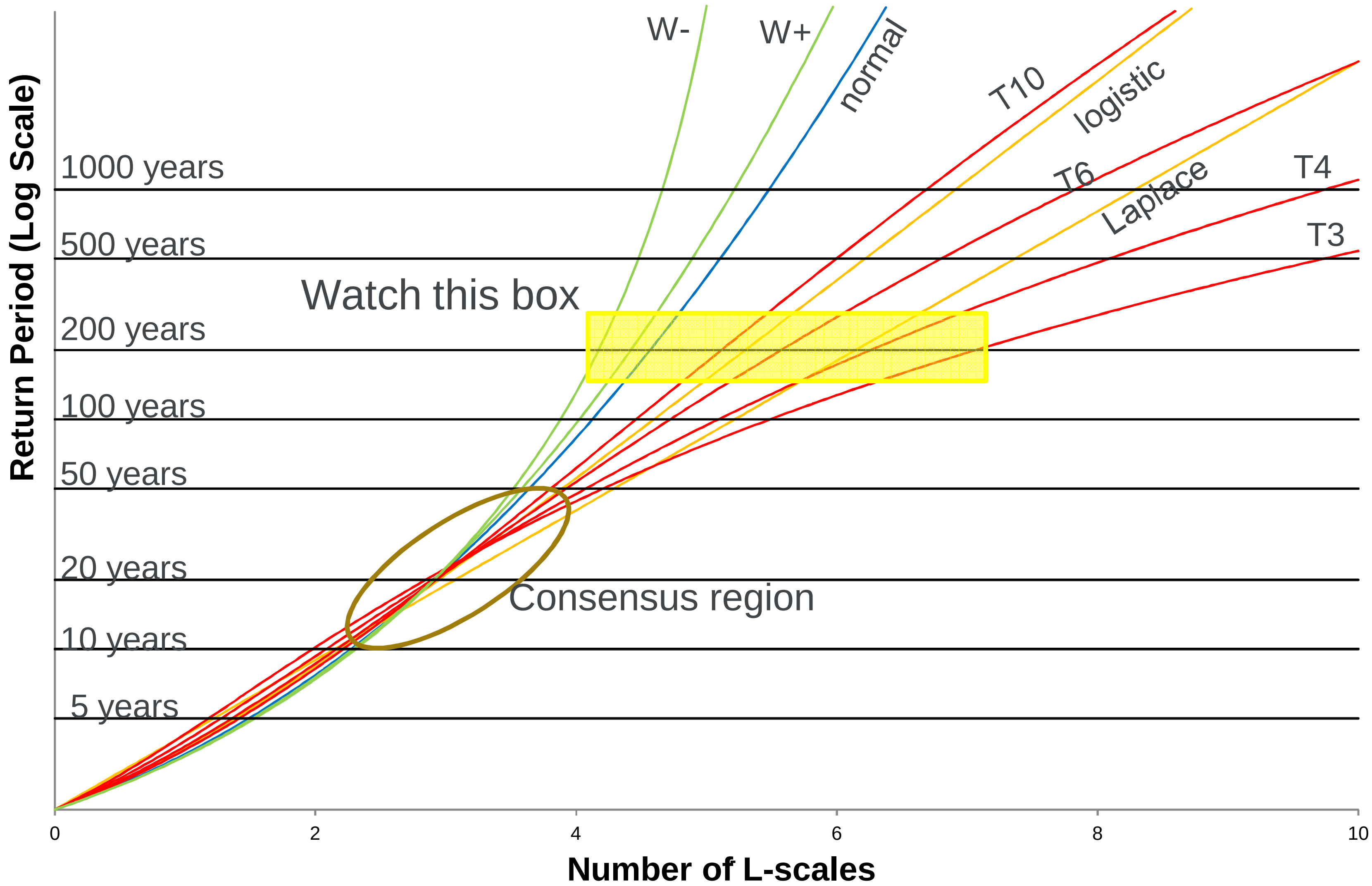
Impact of Model Uncertainty: Using λ_2 as a proxy for Value-at-Risk



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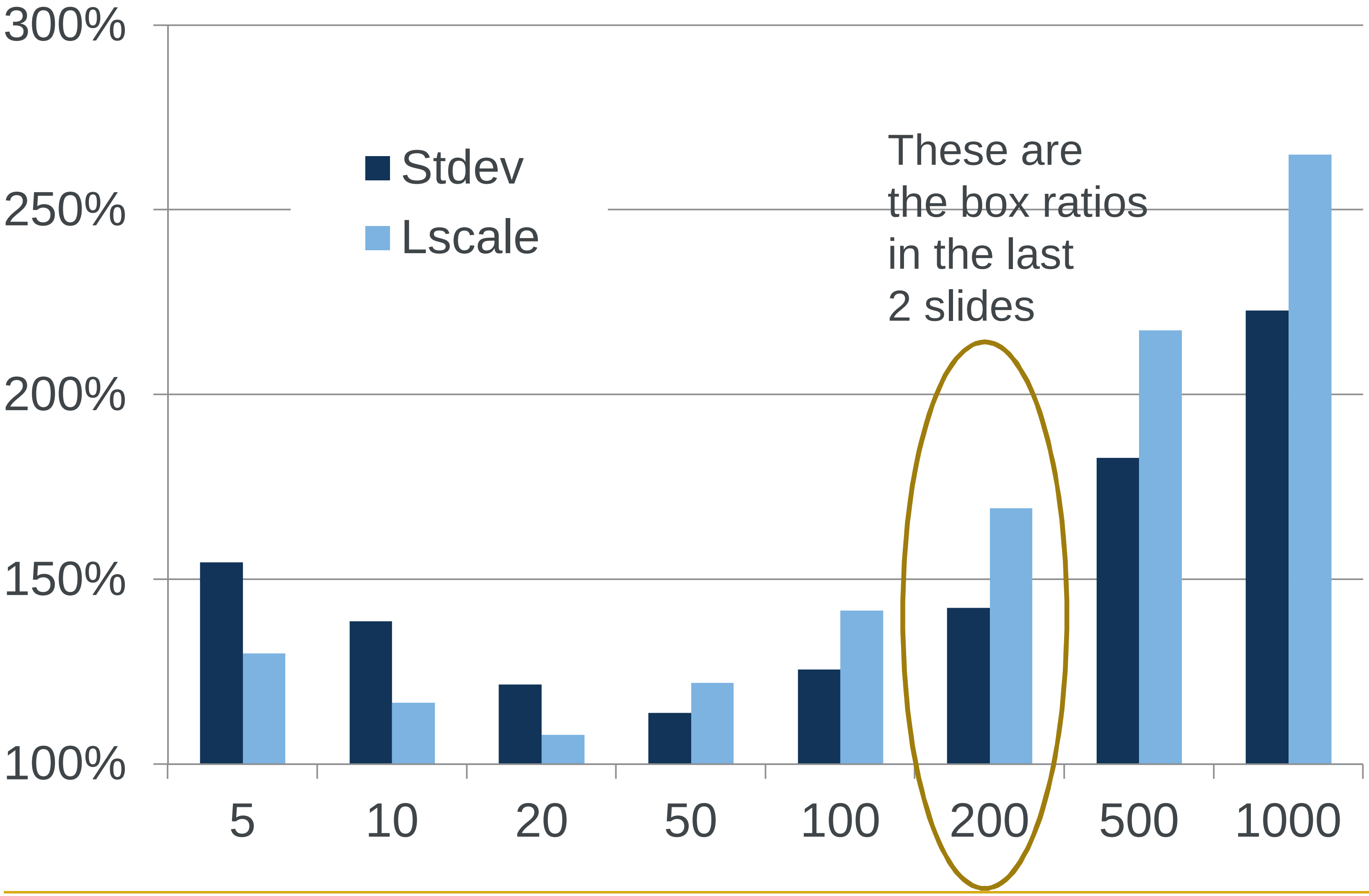
How Dispersion Measure affects Model Risk Ratio of Largest to Smallest by Return Period



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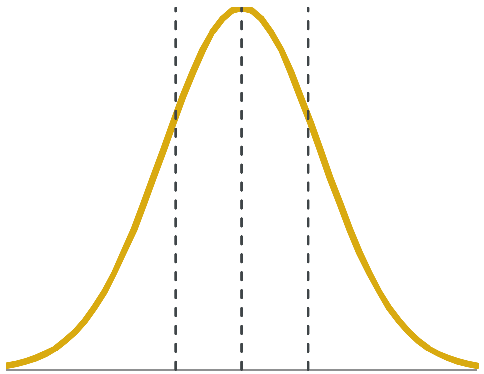
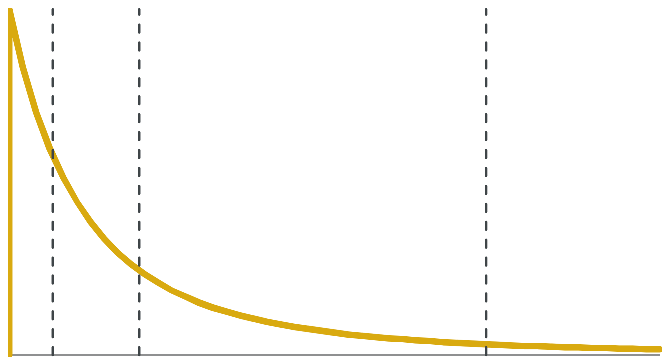
Measuring Skewness

- Dots show **E**min$\{X_1,X_2,X_3\}$, **E**mid$\{X_1,X_2,X_3\}$, **E**max$\{X_1,X_2,X_3\}$
- For standard normal, these are at $-3/2\sqrt{\pi}$, 0, $3/2\sqrt{\pi}$
- For Pareto 2, these are at 0.2, 0.6 and 2.2
- Pareto has positive L-skew as $0.2 + 2.2 > 2 * 0.6$

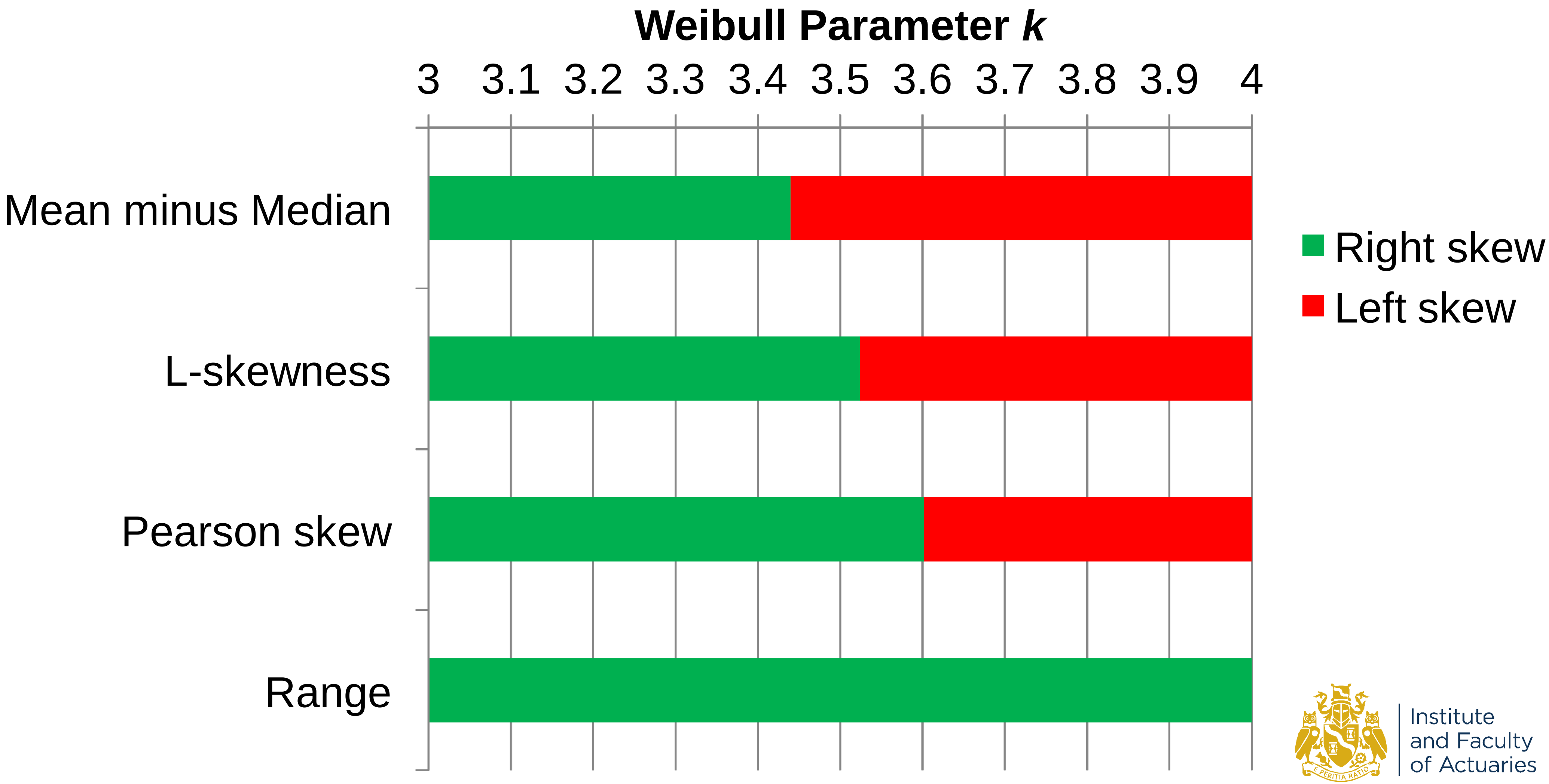


Standard normal density

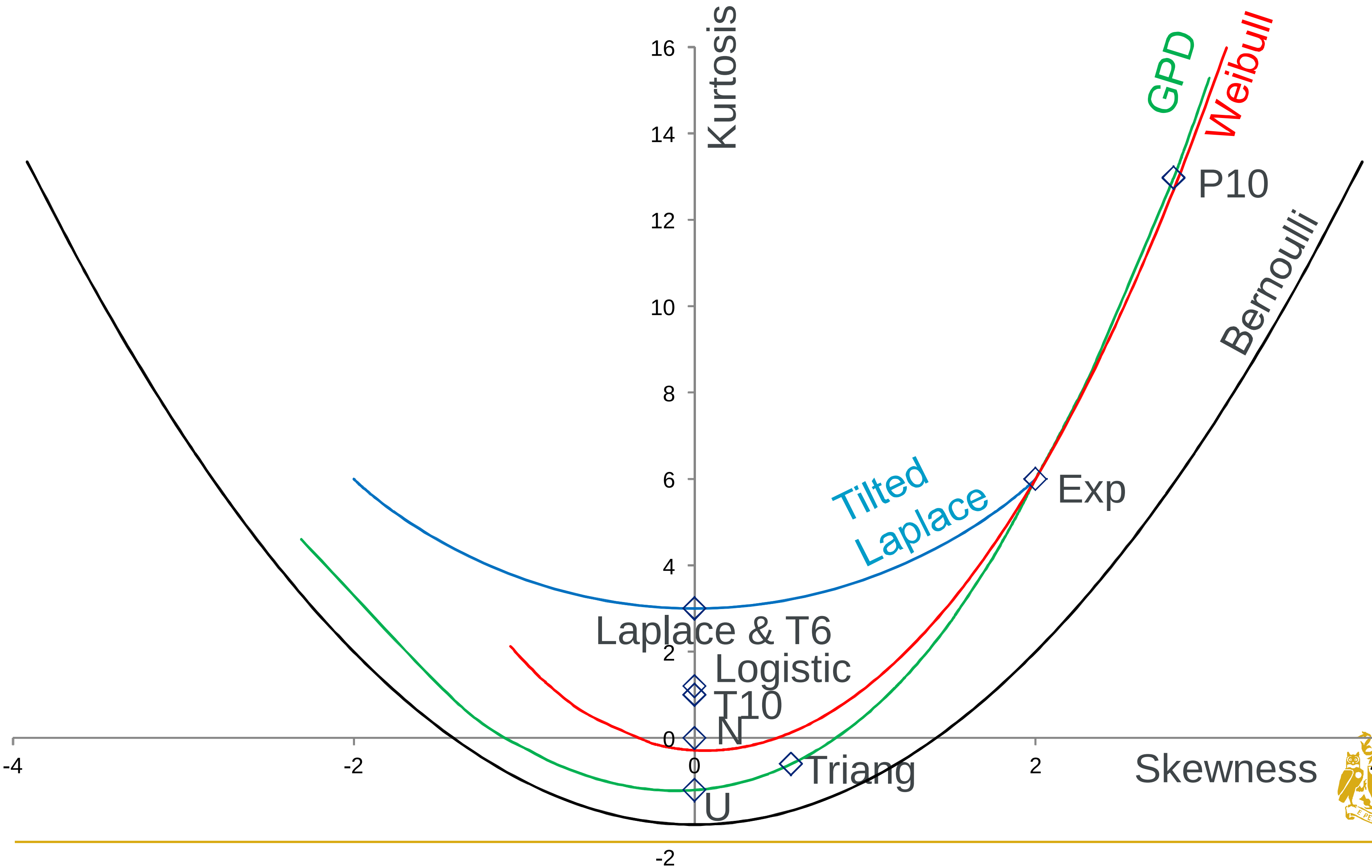


Pareto density (α=2)

Comparing Measures of Skewness
Weibull X where X^k ~ Exponential



Distribution Calibration: Method of Moments



Undefined:
P2, P3, P4
T2, T3, T4
Off the scale
P6

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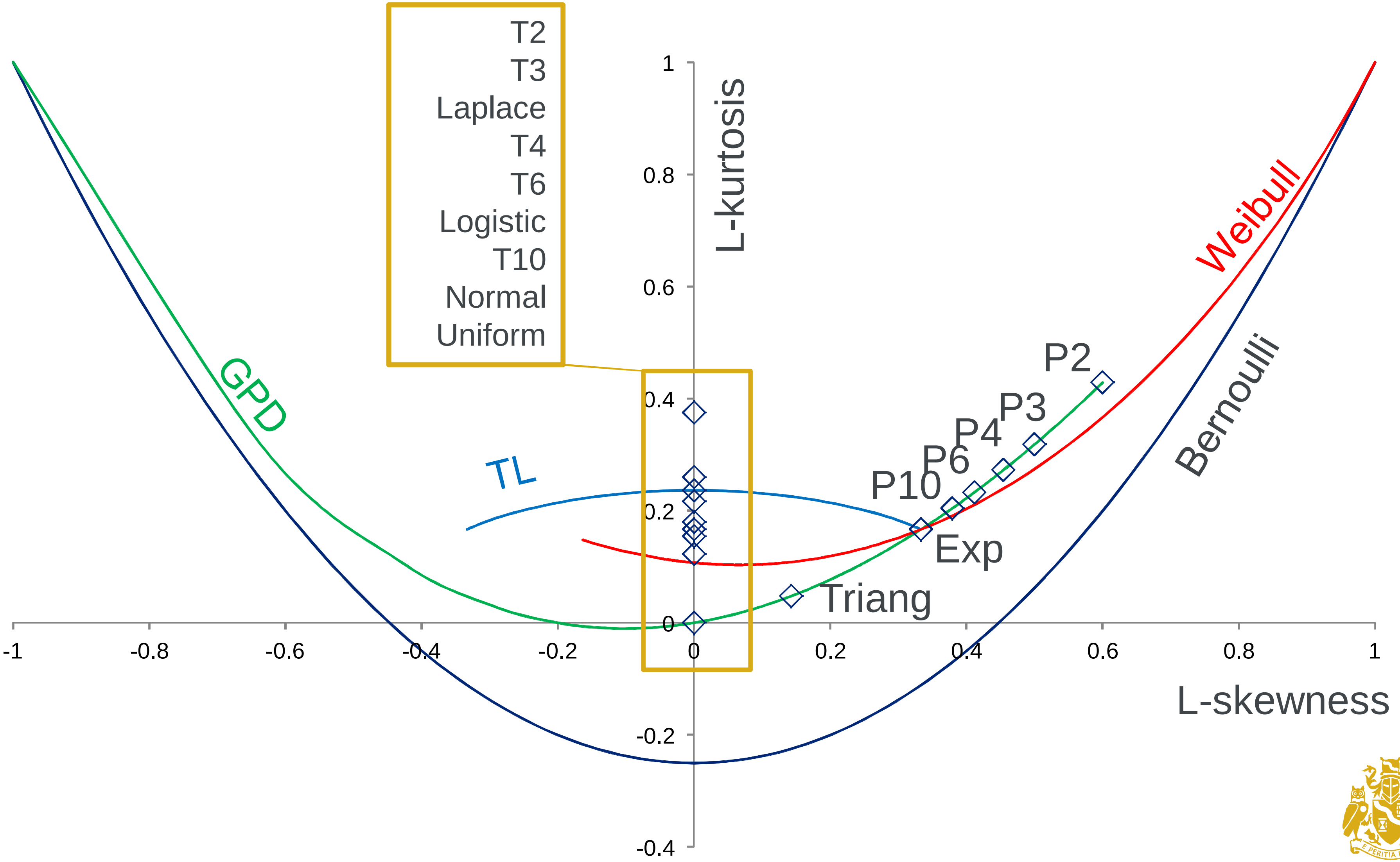
Distribution Calibration: L-moments



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Example Application: Asset Returns

- The simplest, and one of the most widely used asset return models is the geometric random walk. Returns over disjoint periods are independent (and not necessarily normally / lognormally distributed)
- Whether we look at returns in absolute or log terms, we can use mathematical theorems for the Pearson moments or products of random variables, to determine (for example) moments of one-year-returns from behaviour or one-week returns.
- There is no similar (yet known) theorem for L-moments
- But we could calibrate the weekly distribution using L-moments and then convert to Pearson moments (using an assumed distribution) to do the risk aggregation.

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Example Application: Collective Risk Theory

- The Cramér-Lundberg (compound Poisson) collective risk model considers aggregate losses when individual loss amounts are independent observations from a known distribution and the number of losses follows a Poisson distribution, independent of the loss amounts
- There are formulas for the Pearson moments of the aggregate loss distribution given the Poisson frequency and the moments of the individual loss distribution
- There is no similar (yet known) formulas for L-moments
- We could calibrate loss distributions using L-moments, then convert to Pearson moments using an assumed distribution for the risk aggregation.

Example Application: ASRF Credit Model

- Vasiček's Asymptotic Single Risk Factor (ASRF) model is widely used in credit risk modelling, and also forms the basis of the Basel capital accord for regulatory credit risk.
- The model is based on a Gauss copula model, where the two inputs are a probability of default (which turns out to be the mean of the loss distribution) and a copula correlation parameter ρ, applying to all loan pairs.
- The formula's derivation uses an expression for the variances of losses conditional on a single risk factor, which tends to zero for diversified portfolios (Herfindahl index tends to zero).
- There is no similar (yet known) expression using L-moments.
- As the copula drivers cannot be observed directly, the ρ parameter is conventionally calibrated by reference to empirical loss distribution properties.
- The standard deviation of the L-scale are equally suitable for this purpose.

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What about Maximum Likelihood?

- In this session, we have compared classical (Pearson) moments to PWMs.
- More work is required to compare these to alternative methods such as Maximum Likelihood
- Initial results suggest that
  - Max Likelihood has attractive large sample properties if you know the "true" model
  - Practical computational difficulties finding the maxima – the problem often turns out to be unbounded
  - Our methods for examining model mis-specification impact suggest poor resilience, but this is a function of chosen ambiguity set.

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Comparing Methods - Tractability

Pearson Moments

- Analytical formulas known for many familiar distributions (but which came first?)
- Neat proofs known for risk aggregation calculations
- Taught in statistics courses
- Supported in widely-used computer software

Probability Weighted Moments

- Unfamiliar, unsupported, intractable
- Are these barriers cultural or technical?

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Comparing Methods: Statistical

Pearson Moments

- Consistent, but requires higher moments to be finite, excluding members of some distribution families such as Student T and Pareto
- Sampling error is sensitive to outliers
- Relatively good at capturing tails of a distribution
- Multivariate extensions

Probability Weighted Moments

- Finite mean is sufficient for estimation consistency
- Less sensitive to outliers
- Uniquely determine a distribution
- Relatively good at capturing the middle of a distribution

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Questions

Comments

Expressions of individual views by members of the Institute and Faculty of Actuaries and its staff are encouraged.

The views expressed in this presentation are those of the presenters.

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