On the effects of changing mortality patterns on investment, labour supply and consumption under uncertainty

Christian-Oliver Ewald
(Adam Smith Business School, University of Glasgow)

Aihua Zhang
(Department of Mathematics, University of Leicester)

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Agenda

Motivation

In this paper we investigate how changes in mortality rates affect consumption and investment patterns as well as the supply of labour.

If we live longer:

- Do we consume more (pro rata of time) ?
- Do we invest more or less into risky assets ?
- Do we work more or less for longer or shorter times?
- How will this affect GDP?

Motivation

- We build up on the classical life cycle models presented by Merton (1969) and (1971) as well as Bodie, Merton and Samuelson (1992),which was further developped by **Zhang (2010)**, and include **mortality** into our analysis.
  - We study the consumption, labor supply, and portfolio decisions of a representative agent facing age-dependent mortality risk, as presented in UK actuarial life tables.
  - While working, the representative agent receives wage income as well as income from investment into one risky and one risk-free asset.
  - At any time prior to death, the agent can spend his wealth on consumption or further investment and is trying to maximize life time utility from consumption and leisure.

Motivation

- As in Yaari (1965) and Blanchard (1985) we assume the existence of life insurance markets.
  - The difference to Blanchard (1985) is that mortality risk is not assumed to be constant, but in fact obtained from actuarial life tables.
  - Another difference from both, Yaari (1965) and Blanchard (1985), is that we allow for a stochastic investment asset.
  - To the best of our knowledge, our framework is the first in continuous time, where real actuarial life expectancy data can be fed into a stochastic investment and consumption model.

A representative agent's maximization problem

$$\max_{\pi,C,L} \mathbb{E}\left(\int_0^{\tau} e^{-\int_0^t \rho_s ds} u(C_t, L_t) dt\right), \tag{1}$$

where C: instantaneous consumption,

L: instantaneous labour supply,

π: investment choice,

ρ: time preference rate

τ: the time of death with

$$\mathbb{P}\left(\tau \in [t, t+dt) | \tau \geq t\right) = \nu_t dt.$$

The agent's likelihood of surviving until age t is given by

$$\mathbb{P}(\tau > t) = e^{-\int_0^t \nu_s ds}.$$

A representative agent's maximization problem

Assuming the random time τ is **independent** of any of the economic state variables, we can conclude that

$$\mathbb{E}\left(\int_0^\tau e^{-\int_0^t \rho_s ds} u(C_t, L_t)dt\right) = \mathbb{E}\left(\int_0^\infty e^{-\int_0^t \nu_s ds} e^{-\int_0^t \rho_s ds} u(C_t, L_t)dt\right).$$

So the agent's maximization problem (1) can be written as

$$\max_{\pi, C, L} \mathbb{E}\left(\int_0^\infty e^{-\int_0^t \hat{\rho}_s ds} u(C_t, L_t)dt\right), \tag{2}$$

where

$$\hat{\rho}_t = \rho_t + \nu_t$$

The market

⇒ one **risk-less asset** modeled as

$$dB_t = B_t r_t dt$$

⇒ one **risky asset**

$$dS_t = S_t(\mu_t dt + \sigma_t dW_t).$$

⇒ **life insurance** under assumptions:

- competitive insurance market
- free entry and exit

The fair pricing of the insurance contract obliges/entitles the holder to payments per infinitesimal time interval *dt*

$$X_t \nu_t dt,$$

where X_t denotes the current wealth of the agent.

The budget constraint

$$
\begin{aligned}
dX_t &= X_t \{(r_t + \nu_t)dt + \pi_t [(\mu_t - r_t)dt + \sigma_t dW_t]\} \\
&\quad - C_t dt + w_t L_t dt, \\
X_0 &= x \geq 0
\end{aligned}
$$

where w_t is the wage rate.

The market price of risk

$$
\theta_t = \frac{\mu_t - r_t}{\sigma_t} = \frac{\hat{\mu}_t - \hat{r}_t}{\sigma_t}
$$

is unaffected by mortality risk,

where

$$
\begin{aligned}
\hat{r}_t &= r_t + \nu_t \\
\hat{\mu}_t &= \mu_t + \nu_t
\end{aligned}
$$

Stochastic discount factor $\hat{H}_t$

$$\begin{aligned} d\hat{H}_t &= -\hat{H}_t\left(\hat{r}_t dt + \theta_t dW\right) \\ \hat{H}_0 &= 1. \end{aligned}$$

The stochastic discount factor features the mortality adjusted rate $\hat{r}_t$ and the classical market price of risk θ_t.

$$\hat{H}_t = e^{-\int_0^t \left(r_s + \nu_s + \frac{1}{2}\theta_s^2\right) ds - \int_0^t \theta_s dW_s} = e^{-\int_0^t \nu_s ds} H_t,$$

where H_t is the classical stochastic discount factor.

We find:

$$d(\hat{H}_t X_t) = \hat{H}_t X_t\,(\pi_t\sigma_t - \theta_t)\,dW_t - \hat{H}_t C_t dt + \hat{H}_t w_t L_t dt.$$

Transversality condition

$$\lim_{u\to\infty} \mathbb{E}(\hat{H}_u X_u) = 0.$$

Integration gives

$$-\hat{H}_t X_t = \int_t^\infty \hat{H}_s X_s(\pi_s\sigma_s - \theta_s)dW_s - \int_t^\infty \hat{H}_s C_s ds + \int_t^\infty \hat{H}_s w_s L_s ds$$

Taking conditional expectations

$$X_t = \mathbb{E}_t\left[\int_t^\infty \frac{\hat{H}_s}{\hat{H}_t} C_s ds\right] - \mathbb{E}_t\left[\int_t^\infty \frac{\hat{H}_s}{\hat{H}_t} w_s L_s ds\right].$$

At time $t = 0$:

Static budget constraint

$$\mathbb{E}\left(\int_0^\infty \hat{H}_s C_s ds\right) = x + \mathbb{E}\left(\int_0^\infty \hat{H}_s w_s L_s ds\right).$$

Intuition: expected stochastically discounted consumption needs to be equal to initial wealth plus expected stochastically discounted wage income, where the discount factor takes both market risk and mortality risk into account.

Martingale-Based Approach

The agent's maximization problem is equivalent to the following problem:

$$\max_{C,L} \mathbb{E}\left(\int_0^\infty e^{-\int_0^t \hat{\rho}_s ds} u(C_t, L_t) dt \right),$$

subject to

$$\mathbb{E}\left(\int_0^\infty \hat{H}_s C_s ds \right) = x + \mathbb{E}\left(\int_0^\infty \hat{H}_s w_s L_s ds \right).$$

Specification of utility function

$$u(C_t, L_t) := \frac{C_t^{1-\gamma}}{1-\gamma} - b_t \frac{L_t^{1+\eta}}{1+\eta}$$

Intuition: Weigh up benefits from consumptions against dis-benefits from labour in constant relative risk aversion style.

Wage dynamics

$$\frac{dw_t}{w_t} = a_t dt,$$

The optimal consumption and labor supply rate:

$$
\begin{aligned}
C_t^* &= \lambda^{-\frac{1}{\gamma}} e^{-\frac{1}{\gamma}\int_0^t \rho_s ds} H_t^{-\frac{1}{\gamma}} \\
L_t^* &= \lambda^{\frac{1}{\eta}} e^{\frac{1}{\eta}\int_0^t \rho_s ds} (H_t w_t)^{\frac{1}{\eta}} b_t^{-\frac{1}{\eta}}.
\end{aligned}
$$

The non-existence of a bequest motive implies $x = 0$. In this case:

Lagrange multiplier

$$
\lambda = \left(\left(w_0^{\frac{\eta+1}{\eta}} \right) \frac{\int_0^\infty e^{\frac{1}{\eta}\int_0^t \left(\rho_s - (\eta+1)\left(r_s - a_s - \frac{\theta_s^2}{2\eta} \right) \right) ds} \cdot e^{-\int_0^t \nu_s ds} \cdot b_t^{-\frac{1}{\eta}} dt}{\int_0^\infty e^{-\frac{1}{\gamma}\int_0^t \left(\rho_s + (\gamma-1)\left(r_s + \frac{\theta_s^2}{2\gamma} \right) \right) ds} \cdot e^{-\int_0^t \nu_s ds} dt} \right)^{-\frac{\gamma\eta}{\gamma+\eta}}
$$

Euler equation for consumption growth

$$\frac{d}{dt}\mathbb{E}\left(\frac{dC_t^*}{C_t^*}\right) = \frac{1}{\gamma}\left(r_t - \rho_t + \frac{\gamma+1}{2\gamma}\theta_t^2\right).$$

- As expected, the consumption Euler equation does not depend on the mortality risk parameter ν_t.
- The uncertainty attached to the financial market does affect the individual's consumption decision however.

Optimal portfolio strategy

$$\pi_t^* = \frac{1}{\gamma}\frac{\mu_t - r_t}{\sigma_t^2} + g_t \cdot \left(\frac{1}{\gamma} + \frac{1}{\eta}\right)\frac{\mu_t - r_t}{\sigma_t^2} \cdot \frac{w_t L_t^*}{X_t^*}.$$

where g_t is a deterministic function

$$g_t =: \int_t^\infty e^{-\int_t^s \left(\frac{\eta+1}{\eta}\left(r_u - a_u - \frac{\theta_u^2}{2\eta}\right) - \frac{1}{\eta}\rho_u\right)du} \left(\frac{b_s}{b_t}\right)^{-\frac{1}{\eta}} \cdot e^{-\int_t^s \nu_u du} ds.$$

- It represents a modification of the classical Merton (1969) rule $\pi_t = \frac{1}{\gamma}\frac{\mu_t - r_t}{\sigma_t^2}$.
- The agent invests a higher proportion of her/his wealth into the risky asset.
- When mortality rates decrease uniformly, the proportion of wealth invested into the risky asset increases.
- These features can **not** be observed in Yaari (1965) and Blanchard (1985), as these authors only allow for investment in a risk-less asset.

Constant mortality rate

All parameters, including the mortality rate ν_t, are assumed to be constant.

The elasticity of consumption with respect to mortality

$$\frac{\frac{dC_t^*(\nu)}{C_t^*(\nu)}}{\frac{d\nu}{\nu}} = \frac{\left(1 - \frac{\nu + \frac{\rho}{\gamma} + \frac{(\gamma-1)\left(r+\frac{\theta^2}{2\gamma}\right)}{\gamma}}{\left((\nu - \frac{\rho}{\eta} + \frac{(\eta+1)(r-a-\frac{\theta^2}{2\eta}}{\eta} \right)}\right)\eta\nu}{(\gamma+\eta)\left(\nu + \frac{\rho}{\gamma} + \frac{(\gamma-1)\left(r+\frac{\theta^2}{2\gamma}\right)}{\gamma}\right)}$$

Constant mortality rate

Parameter values: $\rho = 0.06$; $\gamma = 2$; $r = 0.03$, $\mu = 0.09$, $\sigma = 0.35$; $a = 0.01$, $b = 0.5$ and $\eta = 3$.

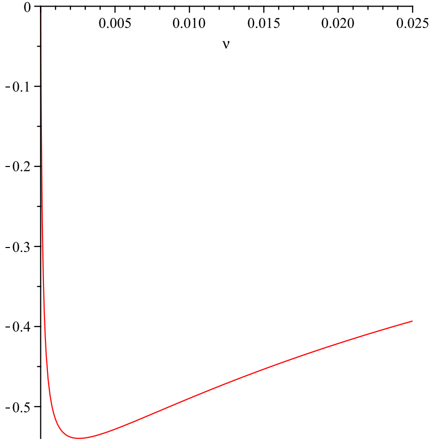

Figure: Elasticity of consumption

Time-dependent mortality rates

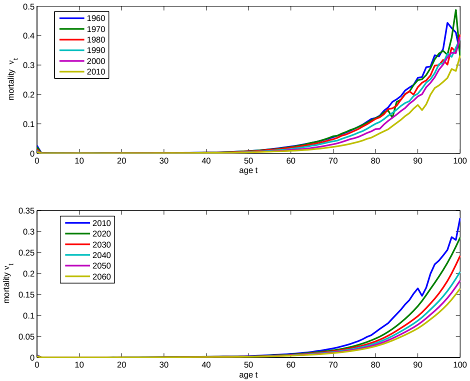

Figure: Age dependent mortality rates for various years between 1960 and 2060 in the UK

Time dependent mortality rates

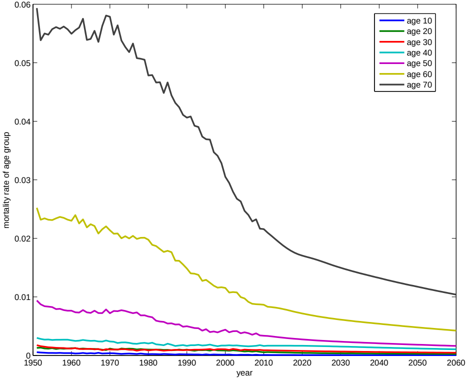

Figure: Mortality rate for selected age groups over the period 1951-2060 in the UK

Time dependent mortality rates

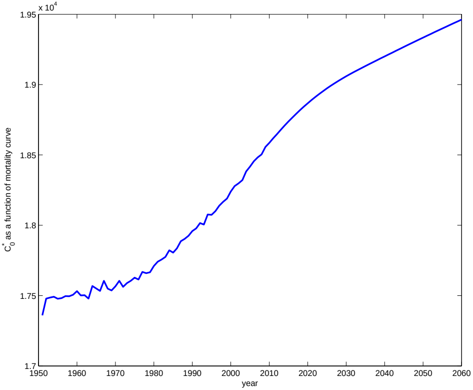

Figure: Consumption under historical mortality at age 25

Time dependent mortality rates

- The figure shows an upward trend, as expected.
- The overall growth in consumption caused by the changing mortality curves over the 110 year period in this case is about 12%.
- From 1980 to 2010, GDP has almost doubled, this can be explained partially by the decline in mortality.

Time dependent mortality rates

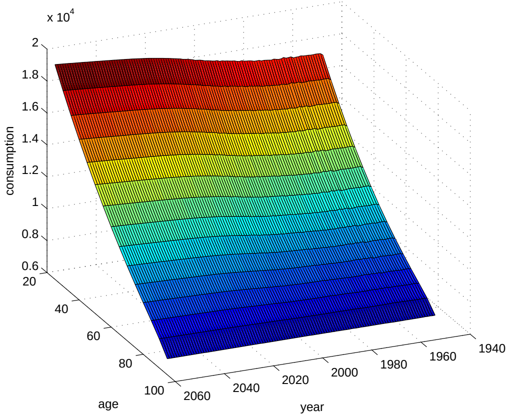

Figure: Consumption under historical mortality at different ages.

Time dependent mortality rates

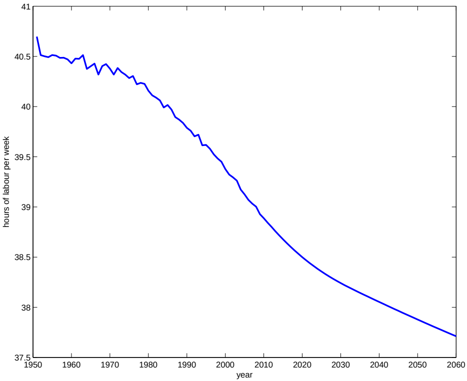

Figure: Labor supply under historical mortality in hours per week

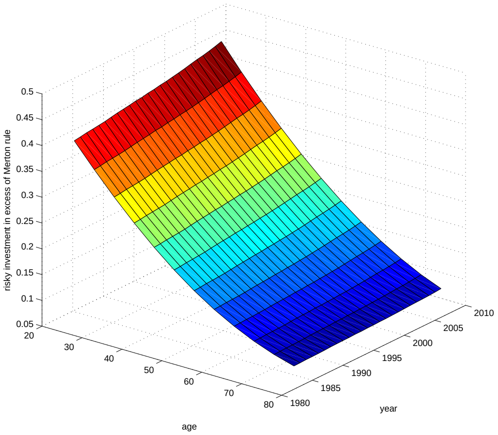

Figure: Excess investment into risky asset as function of age and year.

Examples and Empirical Analysis

- We observe that among all ages of investors between 1980 to 2010 the proportion of wealth invested into the risky asset increases.
- Fixing any year in history between 1980 and 2010, the proportion of wealth invested into the risky asset declines with the age of the investor.
- While this effect can also be observed in reality, it is not present in Yaari (1965), because of the non-existence of a risky asset, and Merton's (1971) or Blanchard's (1985) model, where constant mortality rates are assumed, and a 25 year old investor uses the same portfolio strategy as a 95 year old investor.

Thank You